Applications of some equivalent quasinorms on sequence spaces

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Abstract

Let $x \in \ell_\infty$ be a bounded (real or complex) sequence. We define $a_n(x) = \inf\{\parallel x - \overline{x} \parallel : card(\overline{x}) < n\}, n = 1, 2, ...,$ where $card(\overline{x}) = card\{i \in N : \overline{x_i} \neq 0\}$. Let K the set of all sequences $x \in \ell_\infty$ such that $x_1 \geq x_2 \geq ... \geq 0$ and $card(x) < n < \infty$. The symmetric norming functions, ϕ, of R. Schatten are defined as follows: $\phi : K \to R$ and $\phi(x) > 0$ if $x \neq 0$; $\phi(x + y) \leq \phi(x) + \phi(y)$; $\phi(\alpha x) = \alpha\phi(x), \alpha \geq 0$; $\phi(1, 0, 0, ...) = 1$; $\phi(x) \leq \phi(y)$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, k = 1, 2, ...$

We denote by $\ell_\phi = \{x \in \ell_\infty : \phi(\{a_i(x)\}) < \infty\}$, where $\phi(\{a_i(x)\}) = \lim_{n\to\infty} \phi(a_1(x), a_2(x), ..., a_n(x), 0, 0, ...)$.

We prove that ℓ_ϕ is a quasinormed sequence space, where the quasinorm is $\parallel x \parallel_\phi = \phi(\{a_n(x)\})$.

From the equivalences $\parallel x \parallel_\phi \approx \parallel x \parallel_\phi^+ = \phi(\{a_{2n-1}(x)\})$ and $\parallel x \parallel_{\overline{\phi}} \approx \parallel x \parallel_{\overline{\phi}}^* = \overline{\phi}(\{x_{n^2}\}), \overline{\phi}(x_i) := \phi(\frac{x_i}{i})$ we obtain some interpolation properties for some quasinormed spaces.

1 Introduction

Let ℓ_∞ be the space of all bounded sequences ($x = (x_n) \in \ell_\infty$ if $\parallel x \parallel_\infty = \sup_n | x_n | < \infty$). We denote by $card(x) = card\{n \in N : x_n \neq 0\}$.

For all $x \in \ell_\infty$ we define the sequence of the approximation numbers $(a_n(x))$ as follows:

$$a_n(x) = \inf\{\parallel x - \overline{x} \parallel_\infty : \ card(\overline{x}) < n\}, n = 1, 2, ...$$

It is obvious that:

$$\parallel x \parallel_\infty = a_1(x) \geq a_2(x) \geq ... \geq 0$$

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The subset $K \subseteq \ell_\infty$ is defined as follows:

$$K = \{x \in \ell_\infty : card(x) = n < \infty \; and \; x_1 \geq x_2 \geq ... \geq 0\}$$

A function $\phi : K \to R$ is called symmetric norming function, [3], [4], [6], if the following conditions are verified:

1. ϕ is a norm on the cone K
2. $\phi(1, 0, 0, ...) = 1$
3. If $x, y \in \ell_\infty$ are such that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, k = 1, 2, ...,$ then $\phi(x) \leq \phi(y)$.

Examples of symmetric norming functions are:$\phi_\infty, \phi_\infty(x) = x_1; \phi_p, \phi_p(x) = (\sum x_i^p)^{\frac{1}{p}}, 1 \leq p < \infty$.

For the case $x \in \ell_\infty, x_1 \geq x_2 \geq ... \geq 0$ and $card(x) = \infty$, we take

$$\phi(x) = \lim_{n\to\infty} \phi(x_1, x_2, ..., x_n, 0, 0, ...).$$

Definition 1.1 Let ϕ, ψ be symmetric norming functions. The conjugate (dual) of ψ relative to ϕ is the function:

$$\psi^*_\phi(x) = \sup_{y\in K, y\neq 0} \frac{\phi(xy)}{\psi(y)}, \; where \; xy := (x_1y_1, x_2y_2, ...).$$

Definiton 1.2 $\ell_\phi = \{x \in \ell_\infty : \phi(a_n(x)) < \infty\}$.

We prove that ℓ_ϕ is a quasinormed sequence space.

2 Properties of the numbers $a_n(x)$ and sequence space ℓ_ϕ

For to prove that ℓ_ϕ is a sequence space it is necessary to investigate some properties of $a_n(x)$.

Proposition 2.1 The numbers $a_n(x)$ verify the inequality:

$$\sum_{n=1}^{k} a_n(x_1 + x_2) \leq 2 \sum_{n=1}^{k} (a_n(x_1) + a_n(x_2)), k = 1, 2, ..., x_1, x_2 \in \ell_\infty.$$

Proof. For $\epsilon > 0$ there are $\overline{x}_1, \overline{x}_2 \in \ell_\infty$ such that $card(\overline{x}_i) < n, i = 1, 2,$ and $\| x_i - \overline{x}_i \| \leq a_n(x_i) + \frac{\epsilon}{2}$.

Since $card(\overline{x}_1+\overline{x}_2)<2n-1$, we obtain: $a_{2n-1}(x_1+x_2)\leq\|(x_1+x_2)-(\overline{x}_1+\overline{x}_2)\|\leq a_n(x_1)+a_n(x_2)+\epsilon$. ϵ being arbitrar, it results the inequality: $a_{2n-1}(x_1+x_2)\leq a_n(x_1)+a_n(x_2)$.

Now we can write

$\sum_1^k a_n(x_1+x_2)\leq\sum_{n=1}^{2k}a_n(x_1+x_2)=\sum_1^k a_{2n-1}(x_1+x_2)+\sum_1^k a_{2n}(x_1+x_2)\leq$ $\leq 2\sum_1^k(a_n(x_1)+a_n(x_2)), k=1,2,...$

This prove the inequality (1).

Remark It is known that $a_n(x_1x_2)\leq\|x_1\|_\infty\cdot a_n(x_2)$, [9], [10] and then it results:

For $\lambda\neq 0, a_n(\lambda x)\leq|\lambda|a_n(x)$ and $a_n(x)=a_n(\lambda\cdot\frac{1}{\lambda}x)\leq\frac{1}{|\lambda|}a_n(\lambda x)$.

Hence $a_n(\lambda x)=|\lambda|a_n(x)$, relation which is obvious true for $\lambda=0$.

Now we prove the following

Proposition 2.2 ℓ_ϕ is a quasinormed sequence space with the quasinorm $\|x\|_\phi=\phi(a_n(x))$.

Proof.

1. Obvious $\|x\|_\phi>0$ if $x\neq 0$
2. Let $x_i, i=1,2$, be two seqeuences from ℓ_ϕ. We prove that $x_1+x_2\in\ell_\phi$.

   From proposition 2.1 we obtain:

   $\|x_1+x_2\|_\phi=\phi(\{a_n(x_1+x_2)\})\leq\phi(\{2(a_n(x_1)+a_n(x_2)\})\leq$
   $\leq 2(\|x_1\|_\phi+\|x_2\|_\phi)<\infty$.

   Then $x_1+x_2\in\ell_\phi$ and the property (2) of the quasi-norm $\|\cdot\|_\phi$ is verified.
3. $\|\lambda x\|_\phi=|\lambda|\phi(x)<\infty$ if $\lambda\in R$ and $x\in\ell_\phi$.

3 Equivalent quasinorms on the spaces ℓ_ϕ

Of great interest will be some equivalent quasinorm on the spaces ℓ_ϕ.

We denote: $\|x\|_\phi^+=\phi(a_{2n-1}(x))$ and $\|x\|_\phi^*=\phi(a_{n^2}(x))$.

Proposition 3.1 The quasinorms $\|x\|_\phi^+$ and $\|x\|_\phi$ are equivalent for all ϕ.

Proof.

From the proof of the proposition 2.1 it results that $\|\cdot\|_\phi\approx\|\cdot\|_\phi^+$, where $\|x\|_\phi^+=\phi(\{a_{2n-1}(x)\})$, because:

$$\sum_{n=1}^k a_{2n-1}(x)\leq\sum_{n=1}^k a_n(x)\leq\sum_{n=1}^k a_{2n-1}(x)+\sum_{n=1}^k a_{2n}(x)\leq 2\sum_{n=1}^k a_{2n-1}(x), k=1,2,...$$

and hence

$\| x \|_\phi^+ \leq \| x \|_\phi \leq 2 \| x \|_\phi^+, \forall \phi.$

Proposition 3.2 The quasinorms $\| x \|_\phi$ and $\| x \|_\phi^*$ are equivalent if the function ϕ is $\overline{\phi}$, where $\overline{\phi} : (x_n) \in K \to \phi(\alpha_n x_n), 1 = \alpha_1 \geq \alpha_2 \geq ... \geq 0$ and $\alpha_{n^2} \leq \frac{c}{n}\alpha_n, \forall n \in N$, c being constant.

Proof.

It is obvious that $\| x \|_{\overline{\phi}}^* \leq \| x \|_{\overline{\phi}}$, because $(a_n(x))$ is decreasing.

From the properties of the sequences (α_n) and $(a_n(x))$ we can write:

$$\sum_{n=1}^{k} \alpha_n a_n(x) \leq \sum_{n=1}^{(k+1)^2-1} \alpha_n a_n(x) = \sum_{n=1}^{k} \sum_{j=n^2}^{(n+1)^2-1} \alpha_j a_j(x) \leq 3c \sum_{n=1}^{k} \alpha_n a_{n^2}(x), k = 1, 2, ...$$

Then it results:

$\| x \|_{\overline{\phi}} = \phi(\alpha_n a_n(x)) \leq 3c\phi(\alpha_n a_{n^2}(x)) = 3c \| x \|_{\overline{\phi}}^*$

Hence $\| x \|_{\overline{\phi}} \approx \| x \|_{\overline{\phi}}^*$.

Remark. In a particular case $(\alpha_i) = (\frac{1}{i})$.

4 Applications of the equivalence between $\| x \|_\phi^+$ and $\| x \|_\phi^*$

We denote $X_\Sigma = X_0 + X_1$ and $Y_\Sigma = Y_0 + Y_1$, where (X_0, X_1) and (Y_0, Y_1) are interpolation couples.

If (X_0, X_1) is an interpolation couple of normed spaces, the interpolation space $\overline{X} = (X_0, X_1)_{\theta,q}, 0 < q < \infty, \theta \in (0, 1)$, is defined, [1], [8], [9], [11], as follows:

$$(X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 : (\int_0^\infty [t^{-\theta} K(t, x)]^q \frac{dt}{t})^{\frac{1}{q}} < \infty\},$$

where

$$K(t, x) = \inf_{t>0} \{\| x_0 \|_{X_0} + t \| x_1 \|_{X_1} : x = x_0 + x_1\}$$

Definition 4.1 The interpolation couple (Y_0, Y_1), where Y_0, Y_1 are normed (Banach) spaces, have the approximation property (H) if exists the constant $c > 0$ such that for any $\epsilon > 0$ and any finite sets $Z_i \subset Y_i, i = 1, 2$, exist the application $P \in L(Y_\Sigma, Y_\Sigma)$ for wich $P |_{Y_0} \in L(Y_0, Z_0), P |_{Y_1} \in L(Y_1, Z_1)$ and more the following properties are true:

1. $P(Y_i) \subset Y_0 \cap Y_1$;

2. $\| P \|_{L(Y_i)} \leq c$

3. $\| Px - x \|_{Y_i} \leq \epsilon, \forall x \in Z_i, i = 1, 2$

Let $T : X \to Y$ be a linear and bounded operator ($T \in L(X, Y)$), the dyadic entropy numbers are defined as follows:

$$e_n(T) = \inf\{\sigma > 0 : \exists y_1, ..., y_n \in U_F \text{ such that } TU_X \subseteq \cup_{i=1}^{2^{n-1}} \{y_i + \sigma U_X\},$$
$$\text{where } U_X = \{x \in X : \| x \| \leq 1\}\}$$

In the paper [11] is proved that, if (Y_0, Y_1) have the property (H), the following relation is true:

$$e_{m+n-1}(T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}) \leq 4 \max\{1, c\}[e_m(T : X_0 \to Y_0)^{1-\theta} e_n(T : X_1 \to Y_1)^{\theta}]$$

Where c is the cosntant from the property (H), $m, n = 1, 2, ...$

Corollary 4.1 $e_{2n-1}(T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}) \leq 4 \max\{1, c\}[e_n(T : X_0 \to Y_0)^{1-\theta} e_n(T : X_1 \to Y_1)^{\theta}]$

Let $L_\phi^{(e)}(X, Y)$ be the entropy ideal ($L_\phi^{(e)}(X, Y) = \{T : \phi(\{e_n(T)\}) < \infty\}$) and let $\phi_{(p)}(x_n) = (\phi(x_n^p))^{\frac{1}{p}}, 0 \leq p < \infty$.

From the corollary 4.1 we obtain:

Proposition 4.2 If $(X_0, X_1), (Y_0, Y_1)$ are interpolation couples of normed spaces and (Y_0, Y_0) has the approximation property (H) the following inclusion is true:

$$L^{(e)}_{\psi_{(1-\theta)}}(X_0, Y_0) \cap L^{(e)}_{(\psi_\phi)^*_{(\theta)}}(X_1, Y_1) \subseteq L_\phi^{(e)}((X_0, X_1)_{\theta,q}, (Y_0, Y_1)_{\theta,q}).$$

for all symmetric norming functions ϕ and ψ.

Proof. From the proposition 1.3 and the corollary 4.1 we can write

$$\phi(\{e_n(T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q})\}) \approx$$
$$\approx \phi(\{e_{2n-1}(T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q})\}) \leq$$
$$\leq 4 \max\{1, c\}\phi(e_n(T : X_0 \to Y_0)^{1-\theta} \cdot e_n(T : X_1 \to Y_1)^{\theta}) \leq$$
$$\leq 4 \max\{1, c\}\psi(e_n(T : X_0 \to Y_0)^{1-\theta}) \cdot \psi_\phi^*(e_n(T : X_1 \to Y_1)^{\theta}) < \infty,$$

which proves the inclusion.

Remarks If the particular case of the functions $\phi_p, 1 \leq p < \infty$, we obtain the sequence spaces ℓ_p and the entropy ideals

$$L_p^{(e)}(X, Y) = \{T : (\sum e_n^p(T))^{\frac{1}{p}} < \infty\}.$$

For the ideals $L_p^{(e)}$ it is known the particular inclusion:

$$L_{p_0}^{(e)}(X_0, Y_0) \cap L_{p_1}^{(e)}(X_1, Y_1) \subseteq L_p^{(e)}((X_0, X_1)_{\theta,q}, (Y_0, Y_1)_{\theta,q}),$$

where $1 \leq p_0 < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0, 1)$.

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