

# Compatible partial orders in unary algebras

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## 1. INTRODUCTION

The notion of partial order is well-known in algebra for long time. An important result in the theory of partial orders is the Szpilrajn theorem [3] stating that each partial order can be extended to a linear order. As a consequence we obtain that the maximal partial orders on  $A$  are exactly the linear orders of  $A$ . Let  $f : A \rightarrow A$  be a unary operation. The compatible partial orders of  $(A, f)$  are the partial orders with the following property:  $x \leq_r y$  implies  $f(x) \leq_r f(y)$  for all  $x, y \in A$ , namely  $f$  is an isotone (or order preserving) map on  $A$  [1]. We define the relation  $\sim_f$  and investigate it. Our main result states, that a compatible partial order  $r$  on  $(A, f)$  can always be extended to a compatible  $f$ -quasilinear partial order  $R$  and the maximal compatible partial orders on  $(A, f)$  are exactly the compatible  $f$ -quasilinear partial orders.

## 2. PRELIMINARIES

We consider a partially ordered set or poset as a pair  $(A, \leq_r)$  where  $A$  is a set and  $\leq_r$  is a reflexive, antisymmetric, and transitive binary relation on  $A$ . Let  $(A, \leq_r)$  be a poset and take  $x, y \in A$  with  $x \neq y$ . We say that  $x$  and  $y$  are comparable, when either  $x < y$  or  $y < x$ . Otherwise,  $x$  and  $y$  are incomparable with respect to  $\leq_r$ , denoted  $x \parallel y$  in  $A$ . A poset  $(A, \leq_r)$  is called a chain if every pair of distinct elements from  $A$  is comparable with respect to  $\leq_r$ . When  $(A, \leq_r)$  is a chain, we call  $\leq_r$  a linear order on  $A$ . Similarly, we call a poset an antichain if every pair of distinct elements from  $A$  is incomparable in  $\leq_r$ . If  $f : A \rightarrow A$  is a unary operation, then we can restrict our consideration to the so called compatible partial orders of  $(A, f)$ , i.e. to partial orders with the following property:  $x \leq_r y$  implies  $f(x) \leq_r f(y)$  for all  $x, y \in A$ . In this case the triple  $(A, f, \leq_r)$  is called a partially ordered mono-unary algebra.

**2.1. Definition.** Let  $f : A \rightarrow A$  be a function (unary operation on the set  $A$ ). We define the relation  $\sim_f$  as follows: for  $x, y \in A$  let  $x \sim_f y$  if  $f^k(x) = f^l(y)$  for some integers  $k \geq 0$  and  $l \geq 0$ .

It is straightforward to see that  $\sim_f$  is an equivalence on  $A$ . The equivalence class  $[x]_f$  of an element  $x \in A$  is called the  $f$ -component of  $x$ .  $A / \sim_f$  denotes the set of all equivalence classes of  $\sim_f$ .

**2.2. Example.** Let  $(A, \leq_r)$  be a poset and  $f : A \rightarrow A$  be an unary operation on the set  $A$ . We take  $x, y, z \in A$  as in Fig. 1. We can see  $f^2(y) = f^5(x)$ , so  $x \sim_f y$ , consequently  $[x]_f = [y]_f$ . As well as we can't find integers  $k \geq 0$ ,  $l \geq 0$  such that

$f^k(x) = f^l(z)$ , we have  $x \approx_f z$  and  $y \approx_f z$  too. Clearly,  $A = [x]_f \cup [z]_f$ ,  $[x]_f$  and  $[z]_f$  are  $f$ -components and they give a disjoint cover of  $A$ .

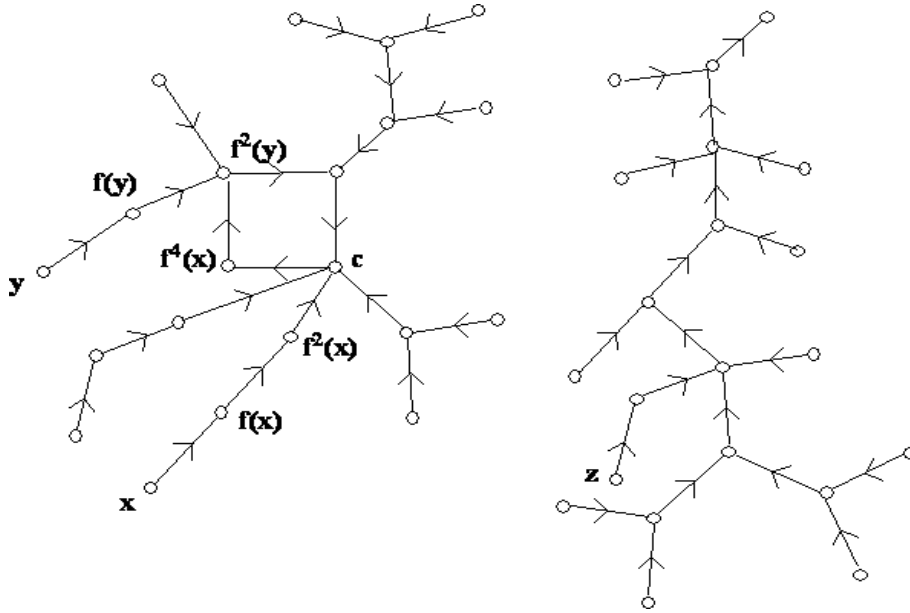


Figure 1.

**2.3. Definition.** An element  $c \in A$  is called *cyclic* with respect to  $f$ , if  $f^m(c) = c$  for some integer  $m \geq 1$ . For a cyclic element

$$n = n(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}$$

is called the *period* of  $c$ . The *cycle*  $C = \{c, f(c), \dots, f^{n-1}(c)\}$  has exactly  $n$  elements and  $f(C) = C$  moreover  $f^k(c) = f^l(c)$  holds if and only if  $k - l$  is divisible by  $n$ .

**2.4. Example.** Let  $(A, \leq_r)$  be a poset and  $f : A \rightarrow A$  be an unary operation on the set  $A$ . We take  $c \in A$  as in Fig. 2.

We can see  $f^5(c) = c$ , so  $n(c) = 5$  and  $C = \{c, f(c), f^2(c), f^3(c), f^4(c)\}$ . In this case  $A$  has five cyclic elements and each element in  $C$  is cyclic of period 5. For example  $f^7(c) = f^2(c)$ , because of  $5 \mid 7 - 2$ .

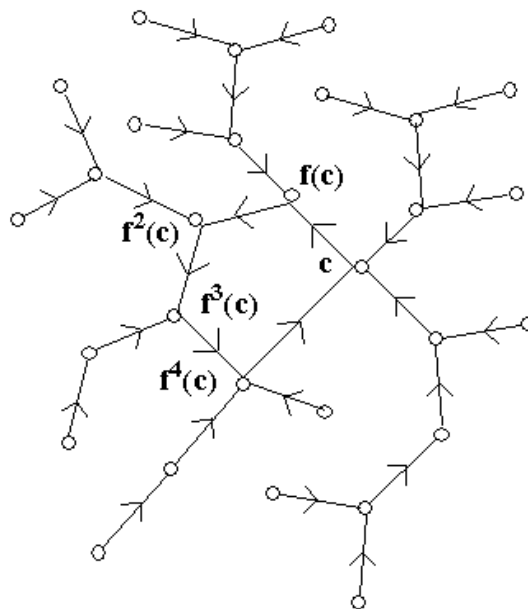


Figure 2.

When  $n(c) = 1$  then  $f(c) = c$  and  $C = \{c\}$ . In this case  $c \in A$  is a fixed point of  $f$ . If  $f : A \rightarrow A$  has a fixed point, then the  $f$ -component of the fixed point has only one cyclic element, this is the fixed point. See Fig. 3.

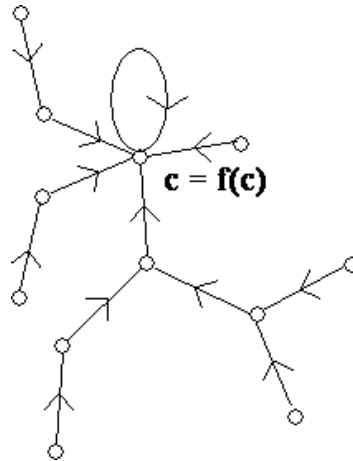


Figure 3.

### 2.5. Proposition.

- All cyclic elements of  $[x]_f$  are in  $C = \{c, f(c), \dots, f^{n-1}(c)\}$  and each element in  $C$  is cyclic of period  $n$ .
- Let  $(A, f, \leq_r)$  is a partially ordered mono-unary algebra. If  $c \in A$  is a cyclic element of period  $n \geq 1$ , then  $C = \{c, f(c), \dots, f^{n-1}(c)\}$  is an antichain with respect to  $\leq_r$ : for  $0 \leq i < j \leq n-1$  the elements  $f^i(c)$  and  $f^j(c)$  are incomparable with respect to  $\leq_r$ , that is  $f^i(c) \parallel f^j(c)$ .

The following definition was introduced by S. Földes and J. Szigeti [4].

**2.6. Definition.** A pair  $(x, y) \in A \times A$  is called  $f$ -prohibited, if we can find integers  $k \geq 0$ ,  $l \geq 0$  and  $m \geq 2$  such that  $m$  is not a divisor of  $k-l$  and  $f^k(x), f^{k+1}(x), \dots, f^{k+m-1}(x)$  are distinct elements, moreover  $f^{k+m}(x) = f^k(x) = f^l(y)$ .

**2.7. Example.** Let  $(A, \leq_r)$  be a poset and  $f : A \rightarrow A$  be an unary operation on the set  $A$ . We take  $x, y \in [x]_f$  as in Fig. 4. We can see  $f^3(x), f^4(x), f^5(x), f^6(x), f^7(x)$  are different and  $f^{3+5}(x) = f^3(x) = f^6(y)$ , and  $5 \nmid 6-3$ , so  $(x, y) \in A \times A$  is an  $f$ -prohibited pair.

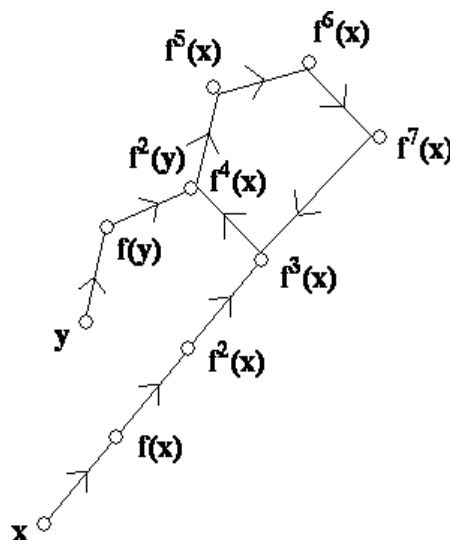


Figure 4.

A pair  $(x, y) \in A \times A$  is  $f$ -prohibited, if and only if  $f^k(x) = f^l(y)$  is cyclic and  $f^{k+l}(x) \neq f^{k+l}(y)$  for some integers  $k \geq 0$  and  $l \geq 0$ . For example  $f^2(y) = f^4(x)$  is cyclic and  $f^{2+4}(x) \neq f^{2+4}(y)$ , so  $(x, y) \in A \times A$  is an  $f$ -prohibited pair.

**2.8. Definition.** Let  $y \in [x]_f$  and  $c \in [x]_f$  a cyclic element of period  $n \geq 1$ . There exists an integer  $t \geq 0$  such that  $f^t(y) = c$ . We denote the distance of  $y$  from  $c$  as follows

$$d(y, c) = \min\{t \mid t \geq 0 \text{ and } f^t(y) = c\}.$$

The following propositions are proved in [4].

**2.9. Proposition.** Let  $(A, f, \leq_r)$  be a partially ordered mono-unary algebra and  $y \in [x]_f$  furthermore  $c \in [x]_f$  a cyclic element of period  $n \geq 1$ . Then we have:

- $(x, y)$  is  $f$ -prohibited if and only if  $n \geq 2$  and  $d(x, c) - d(y, c)$  is not divisible by  $n$ .
- If  $(x, y) \in A \times A$  is an  $f$ -prohibited pair, then  $(x, y) \notin r$  and  $(y, x) \notin r$ , i.e.  $x$  and  $y$  are incomparable elements with respect to  $\leq_r$ , that is  $x \parallel y$ .

### 3. THE ORDER COMPONENTS OF $(A, f, \leq_r)$

**3.1. Definition.** Let  $(A, f, \leq_r)$  be a partially ordered mono-unary algebra. We define the relation  $\triangleleft_r$  on  $B = A / \sim_f = \{[x]_f \mid x \in A\}$  as follows: for  $x, y \in A$  let  $[x]_f \triangleleft_r [y]_f$  if  $x_1 \leq_r y_1$  for some  $x_1 \in [x]_f$  and  $y_1 \in [y]_f$ .

It is easy to see that  $\triangleleft_r$  is a quasiorder on  $B = A / \sim_f$ , namely  $\triangleleft_r$  is reflexive and transitive on  $B$ .

**3.2. Proposition.** If  $[x]_f \triangleleft_r [y]_f$  and  $[y]_f \triangleleft_r [x]_f$  for the  $f$ -components  $[x]_f \neq [y]_f$ , then there is no cyclic element  $c \in [x]_f \cup [y]_f$  of period  $n \geq 1$ .

**3.3. Definition.** The relation  $\equiv_r$  is defined on  $B = A / \sim_f$  as follows: for  $x, y \in A$  let  $[x]_f \equiv_r [y]_f$  if  $[x]_f \triangleleft_r [y]_f$  and  $[y]_f \triangleleft_r [x]_f$ . It is well known, that starting from the quasiorder  $\triangleleft_r$ , the above definition provides an equivalence on  $B$ . We define the order component of  $x$  in  $(A, f, \leq_r)$  by

$$\langle x \rangle = \bigcup_{y \in A \text{ and } [y]_f \equiv_r [x]_f} [y]_f.$$

Clearly,  $[x]_f \subseteq \langle x \rangle \subseteq A$  and  $\langle x \rangle$  is a subalgebra in  $(A, f)$ , which corresponds to the  $\equiv_r$  equivalence class  $[[x]_f]_{\equiv_r}$  of  $[x]_f$  in  $B$ . It is easy to see that  $\{\langle x \rangle \mid x \in A\}$  is a partition of  $A$ :

$$\bigcup_{x \in A} \langle x \rangle = A \text{ and } \langle x \rangle = \langle y \rangle \text{ or } \langle x \rangle \cap \langle y \rangle = \emptyset \text{ for all } x, y \in A.$$

We shall make use of the partial order  $\ll_r$  on  $B / \equiv_r = (A / \sim_f) / \equiv_r$ , which can be derived from  $\triangleleft_r$  in a natural way:  $\langle x \rangle \ll_r \langle y \rangle$  if  $[x]_f \triangleleft_r [y]_f$ .

**3.4. Proposition.** Let  $(A, f, \leq_r)$  be a partially ordered mono-unary algebra. If  $x \in A$  and there is no cyclic element in  $\langle x \rangle$ , then there exists a linear order  $\rho$  on  $\langle x \rangle$  with the following properties:

- $\rho$  is compatible on  $(\langle x \rangle, f)$ :  $(u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho$  for all  $u, v \in \langle x \rangle$ ,
- $\rho$  is an extension of  $\leq_r$  on the elements of  $\langle x \rangle$ .

If  $c \in \langle x \rangle$  is a cyclic element, then  $\langle x \rangle = [x]_f$ .

**3.5. Proposition.** *Let  $(A, f, \leq_r)$  be a partially ordered mono-unary algebra. If  $x \in A$  and  $c \in \langle x \rangle$  is a cyclic element of period  $n \geq 1$ , then there exists a partial order  $\rho$  on  $\langle x \rangle = [x]_f$  with the following properties:*

- $\rho$  is compatible on  $([x]_f, f)$ :  $(u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho$  for all  $u, v \in [x]_f$ ,
- $\rho$  is an extension of  $\leq_r$  on the elements of  $[x]_f$ ,
- $[x]_f$  can be obtained as the union of  $n$  pairwise disjoint chains with respect to  $\rho$ .

## 4. THE MAIN RESULTS

**4.1. Definition.** A compatible partial order  $R$  on a mono-unary algebra  $(A, f)$  is called  $f$ -quasilinear, if either  $(x, y) \in R$  or  $(y, x) \in R$  holds for all non  $f$ -prohibited pairs  $(x, y) \in A \times A$ .

It is easy to see that a compatible  $f$ -quasilinear partial order is linear if and only if the function  $f$  has no proper cycle.

**4.2. Proposition.** *If a compatible partial order  $R$  on a mono-unary algebra  $(A, f)$  is  $f$ -quasilinear, then it is maximal (with respect to containment) among the compatible partial orders of  $(A, f)$ .*

The following theorem was proved by S. Földes and J. Szigeti in [4].

**4.3. Theorem.** *If  $(A, f, \leq_r)$  is a partially ordered mono-unary algebra, then there exists a compatible partial order  $R$  on  $(A, f)$  with the following properties:*

- $R$  is an extension of  $r$ , i.e.  $r \subseteq R$ ,
- $R$  is  $f$ -quasilinear.

**4.4. Corollary.** *A compatible partial order  $R$  on  $(A, f)$  is maximal (with respect to containment) if and only if  $R$  is  $f$ -quasilinear.*

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