Compatible partial orders in unary algebras

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1. Introduction

The notion of partial order is well-known in algebra for long time. An important result in the theory of partial orders is the Szpilrajn theorem [3] stating that each partial order can be extended to a linear order. As a consequence we obtain that the maximal partial orders on A are exactly the linear orders of A. Let $f : A \longrightarrow A$ be a unary operation.The compatible partial orders of (A, f) are the partial orders with the following property: $x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$, namely f is an isotone (or order preserving) map on A [1]. We define the relation $\sim_f$ and investigate it. Our main result states, that a compatible partial order r on (A, f) can always be extended to a compatible f-quasilinear partial order R and the maximal compatible partial orders on (A, f) are exactly the compatible f-quasilinear partial orders.

2. Preliminaries

We consider a partially ordered set or poset as a pair $(A, \leq_r)$ where A is a set and $\leq_r$ is a reflexive, antisymmetric, and transitive binary relation on A. Let $(A, \leq_r)$ be a poset and take $x, y \in A$ with $x \neq y$. We say that x and y are comparable, when either $x < y$ or $y < x$. Otherwise, x and y are incomparable with respect to $\leq_r$, denoted $x \parallel y$ in A. A poset $(A, \leq_r)$ is called a chain if every pair of distinct elements from A is comparable with respect to $\leq_r$. When $(A, \leq_r)$ is a chain, we call $\leq_r$ a linear order on A. Similarly, we call a poset an antichain if every pair of distinct elements from A is incomparable in $\leq_r$. If $f : A \longrightarrow A$ is a unary operation, then we can restrict our consideration to the so called compatible partial orders of (A, f), i.e. to partial orders with the following property: $x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$. In this case the triple $(A, f, \leq_r)$ is called a partially ordered mono-unary algebra.

2.1. Definition. Let $f : A \longrightarrow A$ be a function (unary operation on the set A). We define the relation $\sim_f$ as follows: for $x, y \in A$ let $x \sim_f y$ if $f^k(x) = f^l(y)$ for some integers $k \geq 0$ and $l \geq 0$.
It is straightforward to see that $\sim_f$ is an equivalence on A. The equivalence class $[x]_f$ of an element $x \in A$ is called the *f-component* of x. $A/\sim_f$ denotes the set of all equivalence classes of $\sim_f$.

2.2. Example. Let $(A, \leq_r)$ be a poset and $f : A \longrightarrow A$ be an unary operation on the set A. We take $x, y, z \in A$ as in Fig. 1. We can see $f^2(y) = f^5(x)$, so $x \sim_f y$, consequently $[x]_f = [y]_f$. As well as we can't find integers $k \geq 0$, $l \geq 0$ such that

$f^k(x) = f^l(z)$, we have $x \nsim_f z$ and $y \nsim_f z$ too. Clearly, $A = [x]_f \cup [z]_f$, $[x]_f$ and $[z]_f$ are f-components and they give a disjoint cover of A.

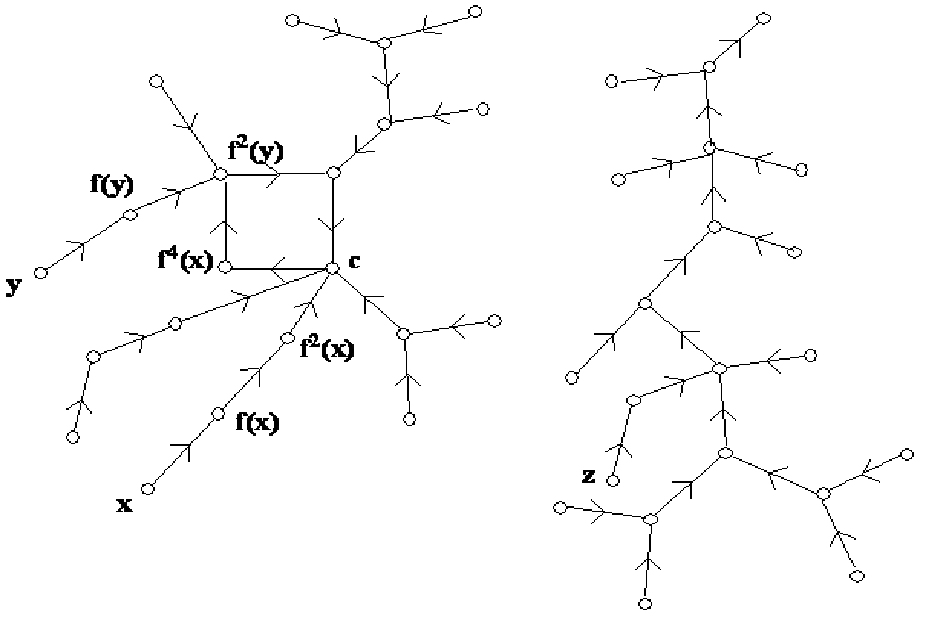

Figure 1.

2.3. Definition. An element $c \in A$ is called *cyclic* with respect to f , if $f^m(c) = c$ for some integer $m \geq 1$. For a cyclic element

$$n = n(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}$$

is called the *period* of c. The *cycle* $C = \{c, f(c), ..., f^{n-1}(c)\}$ has exactly n elements and $f(C) = C$ moreover $f^k(c) = f^l(c)$ holds if and only if $k - l$ is divisible by n.

2.4. Example. Let $(A, \leq_r)$ be a poset and $f : A \longrightarrow A$ be an unary operation on the set A. We take $c \in A$ as in Fig. 2.
We can see $f^5(c) = c$, so $n(c) = 5$ and $C = \{c, f(c), f^2(c), f^3(c), f^4(c)\}$. In this case A has five cyclic elements and each element in C is cyclic of period 5. For example $f^7(c) = f^2(c)$, because of $5 \mid 7 - 2$.

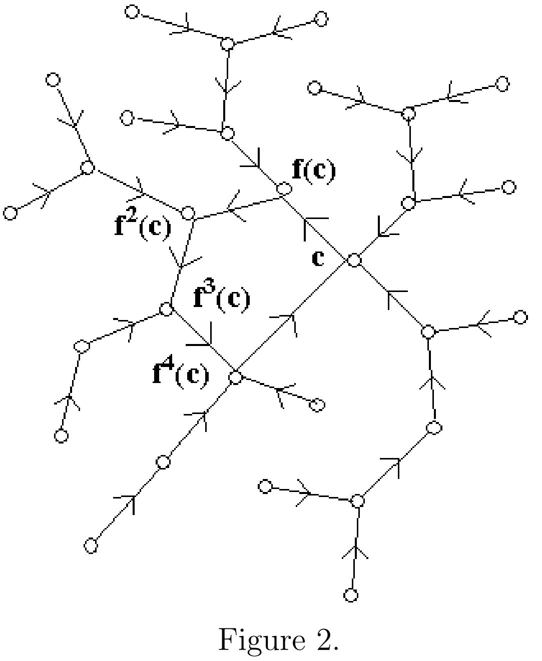

Figure 2.

When $n(c) = 1$ then $f(c) = c$ and $C = \{c\}$. In this case $c \in A$ is a fixed point of f. If $f : A \longrightarrow A$ has a fixed point, then the f-component of the fixed point has only one cyclic element, this is the fixed point. See Fig. 3.

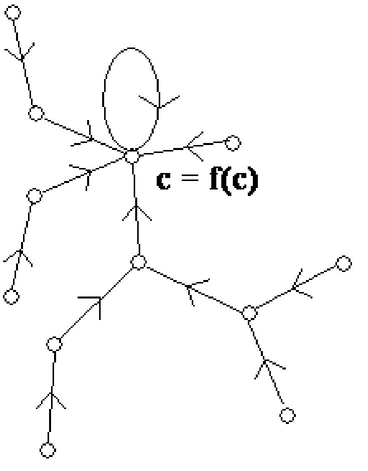

Figure 3.

2.5. Proposition.

- *All cyclic elements of* $[x]_f$ *are in* $C = \{c, f(c), ..., f^{n-1}(c)\}$ *and each element in* C *is cyclic of period* n.
- *Let* $(A, f, \leq_r)$ *is a partially ordered mono-unary algebra. If* $c \in A$ *is a cyclic element of period* $n \geq 1$, *then* $C = \{c, f(c), ..., f^{n-1}(c)\}$ *is an antichain with respect to* $\leq_r$*: for* $0 \leq i < j \leq n-1$ *the elements* $f^i(c)$ *and* $f^j(c)$ *are incomparable with respect to* $\leq_r$, *that is* $f^i(c) \| f^j(c)$.

The following definition was introduced by S. Földes and J. Szigeti [4].

2.6. Definition. A pair $(x, y) \in A \times A$ is called f*-prohibited*, if we can find integers $k \geq 0$, $l \geq 0$ and $m \geq 2$ such that m is not a divisor of $k - l$ and $f^k(x), f^{k+1}(x), ..., f^{k+m-1}(x)$ are distinct elements, moreover $f^{k+m}(x) = f^k(x) = f^l(y)$.

2.7. Example. Let $(A, \leq_r)$ be a poset and $f : A \longrightarrow A$ be an unary operation on the set A. We take $x, y \in [x]_f$ as in Fig. 4. We can see $f^3(x), f^4(x), f^5(x), f^6(x), f^7(x)$ are different and $f^{3+5}(x) = f^3(x) = f^6(y)$, and $5 \nmid 6 - 3$, so $(x, y) \in A \times A$ is an f-prohibited pair.

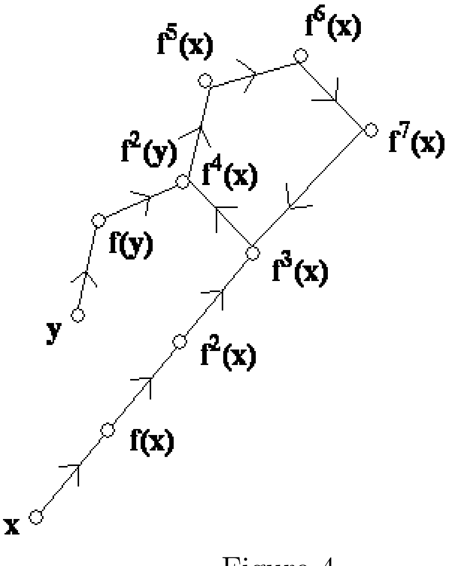

Figure 4.

A pair $(x, y) \in A \times A$ is f-prohibited, if and only if $f^k(x) = f^l(y)$ is cyclic and $f^{k+l}(x) \neq f^{k+l}(y)$ for some integers $k \geq 0$ and $l \geq 0$. For example $f^2(y) = f^4(x)$ is cyclic and $f^{2+4}(x) \neq f^{2+4}(y)$, so $(x, y) \in A \times A$ is an f-prohibited pair.

2.8. Definition. Let $y \in [x]_f$ and $c \in [x]_f$ a cyclic element of period $n \geq 1$. There exists an integer $t \geq 0$ such that $f^t(y) = c$. We denote the distance of y from c as follows

$$d(y, c) = \min\{t \mid t \geq 0 \text{ and } f^t(y) = c\}.$$

The following propositions are proved in [4].

2.9. Proposition. *Let* $(A, f, \leq_r)$ *be a partially ordered mono-unary algebra and* $y \in [x]_f$ *furthermore* $c \in [x]_f$ *a cyclic element of period* $n \geq 1$. Then we have:

- (x, y) *is* f*-prohibited if and only if* $n \geq 2$ *and* $d(x, c) - d(y, c)$ *is not divisible by* n.
- *If* $(x, y) \in A \times A$ *is an* f*-prohibited pair, then* $(x, y) \notin r$ *and* $(y, x) \notin r$*, i.e.* x *and* y *are incomparable elements with respect to* $\leq_r$*, that is* $x \| y$.

3. The order components of $(A, f, \leq_r)$

3.1. Definition. Let $(A, f, \leq_r)$ be a partially ordered mono-unary algebra. We define the relation $\lhd_r$ on $B = A/\sim_f = \{[x]_f \mid x \in A\}$ as follows: for $x, y \in A$ let $[x]_f \lhd_r [y]_f$ if $x_1 \leq_r y_1$ for some $x_1 \in [x]_f$ and $y_1 \in [y]_f$.

It is easy to see that $\lhd_r$ is a quasiorder on $B = A/\sim_f$, namely $\lhd_r$ is reflexive and transitive on B.

3.2. Proposition. *If* $[x]_f \lhd_r [y]_f$ *and* $[y]_f \lhd_r [x]_f$ *for the* f*-components* $[x]_f \neq [y]_f$*, then there is no cyclic element* $c \in [x]_f \cup [y]_f$ *of period* $n \geq 1$.

3.3. Definition. The relation $\equiv_r$ is defined on $B = A/\sim_f$ as follows: for $x, y \in A$ let $[x]_f \equiv_r [y]_f$ if $[x]_f \lhd_r [y]_f$ and $[y]_f \lhd_r [x]_f$. It is well known, that starting from the quasiorder $\lhd_r$, the above definition provides an equivalence on B. We define the order component of x in $(A, f, \leq_r)$ by

$$\langle x \rangle = \bigcup_{y \in A \text{ and } [y]_f \equiv_r [x]_f} [y]_f \,.$$

Clearly, $[x]_f \subseteq \langle x \rangle \subseteq A$ and $\langle x \rangle$ is a subalgebra in (A, f), which corresponds to the $\equiv_r$ equivalence class $[[x]_f]_{\equiv_r}$ of $[x]_f$ in B. It is easy to see that $\{\langle x \rangle \mid x \in A\}$ is a partition of A:

$$\bigcup_{x \in A} \langle x \rangle = A \text{ and } \langle x \rangle = \langle y \rangle \text{ or } \langle x \rangle \cap \langle y \rangle = \varnothing \text{ for all } x, y \in A.$$

We shall make use of the partial order $\ll_r$ on $B/_{\equiv_r} = (A/\sim_f)\big/_{\equiv_r}$, which can be derived from $\lhd_r$ in a natural way: $\langle x \rangle \ll_r \langle y \rangle$ if $[x]_f \lhd_r [y]_f$.

3.4. Proposition. *Let* $(A, f, \leq_r)$ *be a partially ordered mono-unary algebra. If* $x \in A$ *and there is no cyclic element in* $\langle x \rangle$*, then there exists a linear order* ρ *on* $\langle x \rangle$ *with the following properties:*

- ρ *is compatible on* $(\langle x \rangle, f)$*:* $(u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho$ *for all* $u, v \in \langle x \rangle$,
- ρ *is an extension of* $\leq_r$ on the elements of $\langle x \rangle$.

If $c \in \langle x \rangle$ is a cyclic element, then $\langle x \rangle = [x]_f$.

3.5. Proposition. *Let $(A, f, \leq_r)$ be a partially ordered mono-unary algebra. If $x \in A$ and $c \in \langle x \rangle$ is a cyclic element of period $n \geq 1$, then there exists a partial order ρ on $\langle x \rangle = [x]_f$ with the following properties:*

- *ρ is compatible on $([x]_f, f)$: $(u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho$ for all $u, v \in [x]_f$,*
- *ρ is an extension of $\leq_r$ on the elements of $[x]_f$,*
- *$[x]_f$ can be obtained as the union of n pairwise disjoint chains with respect to ρ.*

4. The main results

4.1. Definition. A compatible partial order R on a mono-unary algebra (A, f) is called *f-quasilinear*, if either $(x, y) \in R$ or $(y, x) \in R$ holds for all non f-prohibited pairs $(x, y) \in A \times A$.

It is easy to see that a compatible f-quasilinear partial order is linear if and only if the function f has no proper cycle.

4.2. Proposition. *If a compatible partial order R on a mono-unary algebra (A, f) is f-quasilinear, then it is maximal (with respect to containment) among the compatible partial orders of (A, f).*

The following theorem was proved by S. Földes and J. Szigeti in [4].

4.3. Theorem. *If $(A, f, \leq_r)$ is a partially ordered mono-unary algebra, then there exists a compatible partial order R on (A, f) with the following properties:*

- *R is an extension of r, i.e. $r \subseteq R$,*
- *R is f-quasilinear.*

4.4. Corollary. *A compatible partial order R on (A, f) is maximal (with respect to containment) if and only if R is f-quasilinear.*

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