

# Closedness of the solution mapping for parametric equilibrium problems

M. Bogdan\* and D. Inoan†

## Abstract

The equilibrium problems are treated as classical inequations to obtain closedness of the solution map defined on the set of parameters.

**AMS subject clasification:** 49J93D

**Key words:** Equilibrium problems, Closed set-valued maps, Pseudomonotonicity.

## 1 Introduction

This work is motivated by two recent papers of Bogdan and Kolumbán [2] and Yu et al. [9]. The first one deals with pseudomonotone operators defined on Sobolev spaces and conclude for the closedness of the solution mapping defined on the set of parameters. The second one contains a result on the behavior of Nash equilibrium points.

We start our exposition from the simple remark: if  $f$  is an upper semi-continuous function then the solution set of the inequality  $f(x) \geq 0$  is closed.

Let  $X$  be a Hausdorff topological space and let  $f_n : X \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , be given functions. Let us consider, for  $n \in \mathbb{N}^*$ , the inequations

$$(1)_n \quad f_n(x) \geq 0.$$

Denote by  $S_1(1/n, f_n)$  the solutions set of  $(1)_n$  and by  $S_1(0, f_0)$  the solutions set of

$$(1) \quad f_0(x) \geq 0.$$

Suppose that solutions sets are not empty for all  $n \in \mathbb{N}$ .

Let  $M = \{f : X \rightarrow \mathbb{R} \mid \sup_{x \in X} |f(x)| < +\infty\}$  and let  $\rho$  be the metric on  $M$  given by

$$\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|, \quad f, g \in M.$$

---

\*"Petru Maior" University, ROU - Târgu-Mureş, Romania, e-mail: bgdmarcel@netscape.net

†Department of Mathematics, Technical University of Cluj-Napoca, ROU - Cluj-Napoca, Romania, e-mail: Daniela.Inoan@math.utcluj.ro

We shall denote by  $f_n \xrightarrow{\rho} f_0$  if  $(f_n)_{n \in \mathbb{N}^*}$  converges (uniformly) to  $f_0$  with respect to  $\rho$ .

**Definition 1.** Let  $M'$  be a subset of  $M$ . We say that  $(1)_n$  are stable if the solution mapping  $S_1 : \mathbb{N} \times M' \rightarrow 2^X$  defined above is closed, i.e. if for each sequence  $(x_n)_{n \in \mathbb{N}^*}$ ,  $x_n \in S_1(1/n, f_n)$  with  $x_n \rightarrow x$ ,  $f_n \xrightarrow{\rho} f_0$  as  $n \rightarrow +\infty$  one has  $x \in S_1(0, f_0)$ .

If  $M' = uC(X) = \{f \in M \mid f \text{ is upper semi-continuous on } X\}$  then  $(1)_n$  are stable.

Concerning this property there are several results. Lignola and Morgan established convergence of the solutions of generalized parametric variational inequalities (see [6]). Related to Muu's article [7], more recently, Khanh and Luu [4], Li et al. [5] established *closedness* of the solutions function defined on the set of parameters for the same class of problems, namely parametric vector quasivariational inequalities. They have imposed conditions of upper semi-continuity being motivated by restrictions occurring in economical field. However, their hypotheses are too strong in some applications like variational inequalities governed by differential operators (see [2]).

## 2 Parametric equilibrium problems

Let  $X$  be a normed space,  $D$  a nonempty subset of  $X$  and let  $f_n : X \times X \rightarrow \mathbb{R}$  be given functions. For  $n \in \mathbb{N}^*$  we consider the following parametric equilibrium problem:

(2)<sub>n</sub> Find an element  $x_n \in D$  such that  $f_n(x_n, y) \geq 0, \forall y \in D$ .

Denote by  $S_2(1/n, f_n)$  the solutions set of (2)<sub>n</sub> and by  $S_2(0, f_0)$  the solutions set of the "limit" equilibrium problem:

(2) Find an element  $x_0 \in D$  such that  $f_0(x_0, y) \geq 0, \forall y \in D$ .

Suppose that solutions sets are not empty for all  $n \in \mathbb{N}$ .

To compensate for the lack of upper semi-continuity, Brézis ([3]) introduced the notion of topological pseudomonotonicity (which is a kind of conditioned upper semi-continuity) only in the context of variational inequalities (see [8]).

**Definition 2.** We say that a function  $g : X \times X \rightarrow \mathbb{R}$  is topologically pseudomonotone (in the sense of Brézis) if for each sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \rightharpoonup x$  in  $X$ ,  $\liminf_n g(x_n, x) \geq 0$  imply

$$\limsup_n g(x_n, y) \leq g(x, y), \text{ for all } y \in X.$$

Let  $N = \{f : X \times X \rightarrow \mathbb{R} \mid \sup_{(x,y) \in X \times X} |f(x, y)| < +\infty\}$  and let  $d$  be the metric on  $N$  given by

$$d(f, g) = \sup_{(x,y) \in X \times X} |f(x, y) - g(x, y)|, f, g \in N.$$

We shall denote by  $f_n \xrightarrow{d} f_0$  if  $(f_n)_{n \in \mathbb{N}^*}$  converges (uniformly) to  $f_0$  with respect to  $d$ .

Denote  $pC(X, X) = \{f \in N \mid f \text{ is topologically pseudomonotone}\}$ .

**Proposition 1.** *Let  $(f_n)_{n \in \mathbb{N}^*}$  be a sequence such that  $f_n \in pC(X, X)$ . If  $f_n \xrightarrow{d} f_0$  then  $f_0 \in pC(X, X)$ .*

*Proof.* Let  $\varepsilon > 0$ . Let  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \rightarrow x$  in  $X$ , and  $\liminf_n f_0(x_n, x) \geq 0$ . For  $n$  large we gain  $\liminf_n f_n(x_n, x) \geq 0$ . Since  $f_n$  are in  $pC(X, X)$  one has

$$\limsup_n f_n(x_n, y) \leq f_n(x, y), \text{ for all } y \in X.$$

Thus

$$\limsup_n f_0(x_n, y) \leq f_n(x, y) < f_0(x, y) + \varepsilon, \text{ for all } y \in X.$$

Letting  $\varepsilon \rightarrow 0$  the conclusion follows.  $\square$

**Definition 3.** *Let  $N'$  be a subset of  $N$ . We say that  $(2)_n$  are stable if the solution mapping  $S_2 : \mathbb{N} \times N' \rightarrow 2^X$  defined above is closed, i.e. if for each sequence  $(x_n)_{n \in \mathbb{N}^*}$ ,  $x_n \in S_2(1/n, f_n)$  with  $x_n \rightarrow x_0$ ,  $f_n \xrightarrow{d} f_0$  as  $n \rightarrow +\infty$  one has  $x_0 \in S_2(0, f_0)$ .*

**Proposition 2.** *If  $N' = pC(X, X)$  then  $(2)_n$  are stable.*

*Proof.* The conclusion follows by the following inequalities

$$\begin{aligned} 0 \leq \liminf_n f_n(x_n, y) &\leq \limsup_n f_0(x_n, y) + \liminf_n (f_n(x_n, y) - f_0(x_n, y)) \\ &= \limsup_n f_0(x_n, y) \leq f_0(x, y), \forall y \in D. \end{aligned}$$

$\square$

### 3 Parametric Domains

To this point the feasible set has been "constant". Corresponding to  $(1)_n$  for  $n \in \mathbb{N}^*$  we shall consider the parametric inequations:

$$(1)'_n \quad \text{find } x \in D_n \text{ such that } f_n(x) \geq 0,$$

where  $D_n \subseteq X$  are nonempty and closed.

Denote by  $S'_1(1/n, f_n)$  the solutions set of  $(1)'_n$  and by  $S'_1(0, f_0)$  the solutions set of

$$(1)' \quad \text{find } x \in D_0 \text{ such that } f_0(x) \geq 0, x \in D_0.$$

Suppose that solutions sets are not empty for all  $n \in \mathbb{N}$ . Clearly  $S'_1(1/n, f_n) = f_n^{-1}([0, +\infty)) \cap D_n$  and  $S'_1(0, f_0) = f_0^{-1}([0, +\infty)) \cap D_0$ . If  $\emptyset \neq \bigcap_{n \in \mathbb{N}^*} D_n \subseteq D_0$  we can easily conclude that  $(1)'_n$  are stable in the sense of Definition 1.

Corresponding to  $(2)_n$  we shall consider the parametric equilibrium problems:

$(2)'_n$  Find an element  $x_n \in D_n$  such that  $f_n(x_n, y) \geq 0, \forall y \in D_n$ .

Denote by  $S'_2(1/n, f_n)$  the solutions set of  $(2)'_n$  and by  $S'_2(0, f_0)$  the solutions set of the "limit" equilibrium problem:

$(2)'$  Find an element  $x_0 \in D_0$  such that  $f_0(x_0, y) \geq 0, \forall y \in D_0$ .

Suppose that solutions sets are not empty for all  $n \in \mathbb{N}$ .

**Definition 4.** We say that  $(2)'_n$  are stable if the solution map  $S'_2 : \mathbb{N} \times pC(X, X) \rightarrow 2^X$  is closed, i.e. if  $x_n \in S'_2(1/n, f_n)$ ,  $x_n \rightharpoonup x_0$ , and  $f_n \xrightarrow{d} f_0$  in  $pC(X, X)$  as  $n \rightarrow +\infty$ , then  $x_0 \in S'_2(0, f_0)$ .

We have the following statement.

**Theorem 1.** Let  $X$  be a normed space. Suppose that  $D_n$  are weakly closed for all  $n \in \mathbb{N}^*$  and  $\emptyset \neq \cap_{n \in \mathbb{N}^*} D_n = D_0$ .

Then  $(2)'_n$  are stable in the sense of Definition 4.

*Proof.* Let  $x_n \in D_n$  be such that  $(2)'_n$  is valid. Let  $x_n \rightharpoonup x_0$  in  $X$  thus we have  $x_0 \in D_n$ , so  $\liminf_n f_n(x_n, x_0) \geq 0$ . Since  $f_n \xrightarrow{d} f_0$  it follows

$$\liminf_n f_0(x_n, x_0) \geq 0$$

therefore, by topological pseudomonotonicity of  $f_0$  one has

$$\limsup_n f_0(x_n, y) \leq f_0(x_0, y), \text{ for all } y \in D_0.$$

Finally,

$$\begin{aligned} 0 &\leq \liminf_n f_n(x_n, y) \leq \limsup_n f_0(x_n, y) + \liminf_n (f_n(x_n, y) - f_0(x_n, y)) \leq \\ &\leq \limsup_n f_0(x_n, y) \leq f_0(x_0, y), \forall y \in D_0, \end{aligned}$$

which completes the proof.  $\square$

One question rises. What is the condition for the parametric domains in order to have the stability of  $(2)'_n$ ? We are not naive to consider condition  $\emptyset \neq \cap_{n \in \mathbb{N}^*} D_n = D_0$  a proper one, especially in applications. In [2] we used Mosco's convergence.

## 4 Application to hemivariational inequalities

Let us denote by  $pC_0(X, X) = \{f \in pC(X, X) : f(x, x) = 0, x \in X\}$ . First, we establish that  $pC_0(X, X)$  is stable with respect to addition. The proof is similar to Proposition 2.4 in [8].

**Proposition 3.** If  $f, g \in pC_0(X, X)$  then  $f + g \in pC_0(X, X)$ .

*Proof.* We claim that  $\liminf_n f(x_n, x) \geq 0$  and  $\liminf_n g(x_n, x) \geq 0$ . Otherwise  $\liminf_n g(x_n, x) = -\varepsilon < 0$  and, by passing to a subsequence  $\lim_n g(x_n, x) = -\varepsilon$ . Therefore

$$\begin{aligned} \liminf_n f(x_n, x) &= \liminf_n ((f + g)(x_n, x) - g(x_n, x)) \\ &= \liminf_n (f + g)(x_n, x) - \lim_n g(x_n, x) \geq 0 + \varepsilon. \end{aligned}$$

Since  $f$  is topologically pseudomonotone one has

$$\limsup_n f(x_n, y) \leq f(x, y), \text{ for all } y \in X.$$

Now, put  $x = y$  to reach the contradiction as follows

$$\varepsilon \leq \liminf_n f(x_n, x) \leq \limsup_n f(x_n, x) \leq f(x, x) = 0.$$

The conclusion is obtained from the super-additivity of  $\liminf$  and sub-additivity of  $\limsup$ .  $\square$

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$  with Lipschitz boundary. Let us consider a function  $l : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , such that  $l(x; \cdot, \cdot)$  is upper semi-continuous for a.e.  $x \in \Omega$  and  $l(\cdot; y, z)$  is measurable for all  $y, z \in \mathbb{R}$ . We say that  $l$  satisfies *the growth condition* if there exist  $h_1, h_2 \in \mathbb{R}_+$  such that

$$|l(x; y, z)| \leq (h_1 + h_2|y|)|z|, \text{ a.e. } x \in \Omega, \forall y, z \in \mathbb{R}.$$

**Lemma 1.** ([2]) *If  $l$  satisfies the conditions above, then the function defined on  $H^1(\Omega) \times H^1(\Omega)$  by*

$$(u, w) \mapsto \int_{\Omega} l(x; u(x), w(x)) dx$$

*is weakly upper semi-continuous.*

For  $n \in \mathbb{N}^*$  we consider the following variational inequality problem:

$(VI)_n$  Find an element  $u_n \in K$  such that

$$\begin{aligned} &\int_{\Omega} A_n(x, u_n(x), \nabla u_n(x)) \cdot (\nabla v(x) - \nabla u_n(x)) dx + \\ &\quad + \int_{\Omega} a^n(x, u_n(x), \nabla u_n(x)) (v(x) - u_n(x)) dx + \\ &\quad + \int_{\Omega} l_n(x; u_n(x), v(x) - u_n(x)) dx \geq 0, \forall v \in K, \end{aligned}$$

with  $A_n = (a_1^n, \dots, a_N^n)$ ,  $a^n$ , and  $l_n$  given functions, and  $K$  a closed, convex, nonempty subset of the Sobolev space  $H^1(\Omega)$ .

For a fixed  $n \in \mathbb{N}$ , we consider the operator

$$\mathcal{A}_n : H^1(\Omega) \rightarrow (H^1(\Omega))^*, \quad \mathcal{A}_n(u) = \mathcal{B}_n(u, u),$$

where

$$\mathcal{B}_n(u, v)(w) = \int_{\Omega} \left\{ \sum_{i=1}^N a_i^n(x, u(x), \nabla u(x)) \cdot \partial_i w(x) \right\} dx + \int_{\Omega} a^n(x, u(x), \nabla u(x)) w(x) dx.$$

The operators  $\mathcal{A}_n$  are topologically pseudomonotone if satisfies some reasonable conditions (see [8], page 76; [2]).

$(VI)_n$  can be written as

$$\text{find } u_n \in K \text{ such that } f_n(u_n, v) + L_n(u_n, v) \geq 0, \forall v \in K,$$

$$\text{where } f_n(u, v) = \langle \mathcal{A}_n(u), v - u \rangle \text{ and } L_n(u, v) = \int_{\Omega} l_n(x; u(x), v(x) - u(x)) dx.$$

Finally, we can apply Proposition 2 and Proposition 3 to obtain a result on the stability of  $(VI)_n$ .

## References

- [1] Anh, L. Q., and Khanh, P. Q.: *Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces*, Journal of Global Optimization, original paper, DOI 10.1007/s10898-006-9062-8 (2006).
- [2] Bogdan, M., and Kolumbán, J.: *On Nonlinear Parametric Variational Inequalities*, Nonlinear Analysis, Vol. 67, No. 7, pp. 2272 - 2282, 2007.
- [3] Brézis, H.: *Équations et inéquations non linéaires en dualité*, Annales de l'Institut Fourier (Grenoble) 18(1)(1965), pp. 115 - 175.
- [4] Khanh, P. Q., and Luu, L. M.: *Upper Semicontinuity of the Solution Set to Parametric Vector Quasivariational Inequalities*, Journal of Global Optimization, **32**(2005), 569 - 580.
- [5] Li, S. J., Chen, G. Y., and Teo, K. L.: *On the Stability of Generalized Vector Quasivariational Inequality Problems*, Journal of Optimization Theory and Applications, vol. 107, no. 1, pp. 35 - 50, 2000.
- [6] Lignola, M. B., and Morgan, J.: *Generalized Variational Inequalities with Pseudomonotone Operators Under Perturbations*, Journal of Optimization Theory and Applications, Vol. 101, No. 1, pp. 213 - 220, 1999.
- [7] Muu, L. D.: *Stability property of a class of variational inequalities*, Mathematische Operationforschung und Statistik, Serie Optimization 15, pp. 347 - 351, 1984.
- [8] Showalter, R. E.: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, vol. 49, 1997.
- [9] Yu, J., Yang, H., and Yu, C.: *Well-posed Ky Fan's point, quasi-variational inequality and Nash equilibrium problems*, Nonlinear Analysis, Vol. 66, pp. 777 - 790, 2007.