# Closedness of the solution mapping for parametric equilibrium problems

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**Abstract**

The equilibrium problems are treated as classical inequations to obtain closedness of the solution map defined on the set of parameters.

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**Key words:** Equilibrium problems, Closed set-valued maps, Pseudomonotonicity.

# 1 Introduction

This work is motivated by two recent papers of Bogdan and Kolumbán [2] and Yu et al. [9]. The first one deals with pseudomonotone operators defined on Sobolev spaces and conclude for the closedness of the solution mapping defined on the set of parameters. The second one contains a result on the behavior of Nash equilibrium points.

We start our exposition from the simple remark: if $f$ is an upper semi-continuous function then the solution set of the inequality $f(x) \geq 0$ is closed.

Let $X$ be a Hausdorff topological space and let $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be given functions. Let us consider, for $n \in \mathbb{N}^*$, the inequations

$$(1)_n \qquad f_n(x) \geq 0.$$

Denote by $S_1(1/n, f_n)$ the solutions set of $(1)_n$ and by $S_1(0, f_0)$ the solutions set of

$$(1) \qquad f_0(x) \geq 0.$$

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

Let $M = \{f : X \to \mathbb{R} \mid \sup_{x\in X} |f(x)| < +\infty\}$ and let $\rho$ be the metric on $M$ given by

$$\rho(f, g) = \sup_{x\in X} |f(x) - g(x)|,\ f, g \in M.$$

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We shall denote by $f_n \xrightarrow{\rho} f_0$ if $(f_n)_{n\in\mathbb{N}^*}$ converges (uniformly) to $f_0$ with respect to $\rho$.

**Definition 1.** *Let $M'$ be a subset of $M$. We say that $(1)_n$ are stable if the solution mapping $S_1 : \mathbb{N}\times M' \to 2^X$ defined above is closed, i.e. if for each sequence $(x_n)_{n\in\mathbb{N}^*}$, $x_n \in S_1(1/n, f_n)$ with $x_n \to x$, $f_n \xrightarrow{\rho} f_0$ as $n \to +\infty$ one has $x \in S_1(0, f_0)$.*

If $M' = uC(X) = \{f \in M \mid f \text{ is upper semi-continuous on } X\}$ then $(1)_n$ are stable.

Concerning this property there are several results. Lignola and Morgan established convergence of the solutions of generalized parametric variational inequalities (see [6]). Related to Muu's article [7], more recently, Khanh and Luu [4], Li et al. [5] established *closedness* of the solutions function defined on the set of parameters for the same class of problems, namely parametric vector quasivariational inequalities. They have imposed conditions of upper semi-continuity being motivated by restrictions occurring in economical field. However, their hypotheses are too strong in some applications like variational inequalities governed by differential operators (see [2]).

# 2 Parametric equilibrium problems

Let $X$ be a normed space, $D$ a nonempty subset of $X$ and let $f_n : X \times X \to \mathbb{R}$ be given functions. For $n \in \mathbb{N}^*$ we consider the following parametric equilibrium problem:

$$(2)_n \;\; \text{Find an element } x_n \in D \text{ such that } f_n(x_n, y) \geq 0, \forall\, y \in D.$$

Denote by $S_2(1/n, f_n)$ the solutions set of $(2)_n$ and by $S_2(0, f_0)$ the solutions set of the "limit" equilibrium problem:

$$(2) \;\; \text{Find an element } x_0 \in D \text{ such that } f_0(x_0, y) \geq 0, \forall\, y \in D.$$

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

To compensate for the lack of upper semi-continuity, Brézis ([3]) introduced the notion of topological pseudomonotonicity (which is a kind of conditioned upper semi-continuity) only in the context of variational inequalities (see [8]).

**Definition 2.** *We say that a function $g : X \times X \to \mathbb{R}$ is* topologically pseudomonotone *(in the sense of Brézis) if for each sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \rightharpoonup x$ in $X$, $\liminf_n g(x_n, x) \geq 0$ imply*

$$\limsup_n g(x_n, y) \leq g(x, y), \textit{ for all } y \in X.$$

Let $N = \{f : X \times X \to \mathbb{R} \mid \sup_{(x,y)\in X\times X} |f(x, y)| < +\infty\}$ and let $d$ be the metric on $N$ given by

$$d(f, g) = \sup_{(x,y)\in X\times X} |f(x, y) - g(x, y)|,\; f, g \in N.$$

We shall denote by $f_n \xrightarrow{d} f_0$ if $(f_n)_{n\in\mathbb{N}^*}$ converges (uniformly) to $f_0$ with respect to $d$.

Denote $pC(X,X) = \{f \in N \mid f \text{ is topologically pseudomonotone }\}$.

**Proposition 1.** *Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence such that $f_n \in pC(X,X)$. If $f_n \xrightarrow{d} f_0$ then $f_0 \in pC(X,X)$.*

*Proof.* Let $\varepsilon > 0$. Let $(x_n)_{n\in\mathbb{N}}$ with $x_n \rightharpoonup x$ in $X$, and $\liminf_n f_0(x_n,x) \geq 0$. For $n$ large we gain $\liminf_n f_n(x_n,x) \geq 0$. Since $f_n$ are in $pC(X,X)$ one has

$$\limsup_n f_n(x_n,y) \leq f_n(x,y), \text{ for all } y \in X.$$

Thus

$$\limsup_n f_0(x_n,y) \leq f_n(x,y) < f_0(x,y) + \varepsilon, \text{ for all } y \in X.$$

Letting $\varepsilon \to 0$ the conclusion follows. □

**Definition 3.** *Let $N'$ be a subset of $N$. We say that $(2)_n$ are stable if the solution mapping $S_2 : \mathbb{N}\times N' \to 2^X$ defined above is closed, i.e. if for each sequence $(x_n)_{n\in\mathbb{N}^*}$, $x_n \in S_2(1/n, f_n)$ with $x_n \rightharpoonup x_0$, $f_n \xrightarrow{d} f_0$ as $n \to +\infty$ one has $x_0 \in S_2(0, f_0)$.*

**Proposition 2.** *If $N' = pC(X,X)$ then $(2)_n$ are stable.*

*Proof.* The conclusion follows by the following inequalities

$$\begin{aligned} 0 \leq \liminf_n f_n(x_n,y) &\leq \limsup_n f_0(x_n,y) + \liminf_n \big(f_n(x_n,y) - f_0(x_n,y)\big) \\ &= \limsup_n f_0(x_n,y) \leq f_0(x,y), \ \forall\, y \in D. \end{aligned}$$

□

# 3 Parametric Domains

To this point the feasible set has been "constant". Corresponding to $(1)_n$ for $n \in \mathbb{N}^*$ we shall consider the parametric inequations:

$$(1)'_n \qquad\qquad \text{find } x \in D_n \text{ such that } f_n(x) \geq 0,$$

where $D_n \subseteq X$ are nonempty and closed.

Denote by $S'_1(1/n, f_n)$ the solutions set of $(1)'_n$ and by $S'_1(0, f_0)$ the solutions set of

$$(1)' \qquad\qquad \text{find } x \in D_0 \text{ such that } f_0(x) \geq 0, x \in D_0.$$

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$. Clearly $S'_1(1/n, f_n) = f_n^{-1}([0,+\infty)) \cap D_n$ and $S'_1(0, f_0) = f_0^{-1}([0,+\infty)) \cap D_0$. If $\emptyset \neq \cap_{n\in\mathbb{N}^*} D_n \subseteq D_0$ we can easily conclude that $(1)'_n$ are stable in the sense of Definition 1.

Corresponding to $(2)_n$ we shall consider the parametric equilibrium problems:

$(2)'_n$ Find an element $x_n \in D_n$ such that $f_n(x_n, y) \geq 0, \forall\, y \in D_n$.

Denote by $S'_2(1/n, f_n)$ the solutions set of $(2)'_n$ and by $S'_2(0, f_0)$ the solutions set of the "limit" equilibrium problem:

$(2)'$ Find an element $x_0 \in D_0$ such that $f_0(x_0, y) \geq 0, \forall\, y \in D_0$.

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

**Definition 4.** *We say that $(2)'_n$ are stable if the solution map $S'_2 : \mathbb{N} \times pC(X, X) \to 2^X$ is closed, i.e. if $x_n \in S'_2(1/n, f_n)$, $x_n \rightharpoonup x_0$, and $f_n \xrightarrow{d} f_0$ in $pC(X, X)$ as $n \to +\infty$, then $x_0 \in S'_2(0, f_0)$.*

We have the following statement.

**Theorem 1.** *Let $X$ be a normed space. Suppose that $D_n$ are weakly closed for all $n \in \mathbb{N}^*$ and $\emptyset \neq \cap_{n\in\mathbb{N}^*} D_n = D_0$.*

*Then $(2)'_n$ are stable in the sense of Definition 4.*

*Proof.* Let $x_n \in D_n$ be such that $(2)'_n$ is valid. Let $x_n \rightharpoonup x_0$ in $X$ thus we have $x_0 \in D_n$, so $\liminf_n f_n(x_n, x_0) \geq 0$. Since $f_n \xrightarrow{d} f_0$ it follows

$$\liminf_n f_0(x_n, x_0) \geq 0$$

therefore, by topological pseudomonotonicity of $f_0$ one has

$$\limsup_n f_0(x_n, y) \leq f_0(x_0, y), \text{ for all } y \in D_0.$$

Finally,

$$\begin{aligned} 0 \leq \liminf_n f_n(x_n, y) &\leq \limsup_n f_0(x_n, y) + \liminf_n \big(f_n(x_n, y) - f_0(x_n, y)\big) \leq \\ &\leq \limsup_n f_0(x_n, y) \leq f_0(x_0, y), \,\forall\, y \in D_0, \end{aligned}$$

which completes the proof. □

One question rises. What is the condition for the parametric domains in order to have the stability of $(2)'_n$ ? We are not naive to consider condition $\emptyset \neq \cap_{n\in\mathbb{N}^*} D_n = D_0$ a proper one, especially in applications. In [2] we used Mosco's convergence.

# 4 Application to hemivariational inequalities

Let us denote by $pC_0(X, X) = \{f \in pC(X, X) : f(x, x) = 0, x \in X\}$. First, we establish that $pC_0(X, X)$ is stable with respect to addition. The proof is similar to Proposition 2.4 in [8].

**Proposition 3.** *If $f, g \in pC_0(X, X)$ then $f + g \in pC_0(X, X)$.*

*Proof.* We claim that $\liminf_n f(x_n, x) \geq 0$ and $\liminf_n g(x_n, x) \geq 0$. Otherwise $\liminf_n g(x_n, x) = -\varepsilon < 0$ and, by passing to a subsequence $\lim_n g(x_n, x) = -\varepsilon$. Therefore

$$\begin{aligned} \liminf_n f(x_n, x) &= \liminf_n \big((f+g)(x_n, x) - g(x_n, x)\big) \\ &= \liminf_n (f+g)(x_n, x) - \lim_n g(x_n, x) \geq 0 + \varepsilon. \end{aligned}$$

Since $f$ is topologically pseudomonotone one has

$$\limsup_n f(x_n, y) \leq f(x, y), \text{ for all } y \in X.$$

Now, put $x = y$ to reach the contradiction as follows

$$\varepsilon \leq \liminf_n f(x_n, x) \leq \limsup_n f(x_n, x) \leq f(x, x) = 0.$$

The conclusion is obtained from the super-additivity of $\liminf$ and sub-additivity of $\limsup$. $\square$

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary. Let us consider a function $l : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that $l(x; \cdot, \cdot)$ is upper semi-continuous for a.e. $x \in \Omega$ and $l(\cdot; y, z)$ is measurable for all $y, z \in \mathbb{R}$. We say that $l$ satisfies *the growth condition* if there exist $h_1, h_2 \in \mathbb{R}_+$ such that

$$|l(x; y, z)| \leq (h_1 + h_2|y|)|z|, \text{ a.e. } x \in \Omega, \forall\, y, z \in \mathbb{R}.$$

**Lemma 1.** ([2]) *If $l$ satisfies the conditions above, then the function defined on $H^1(\Omega) \times H^1(\Omega)$ by*

$$(u, w) \rightarrowtail \int_\Omega l\big(x; u(x), w(x)\big)\, dx$$

*is weakly upper semi-continuous.*

For $n \in \mathbb{N}^*$ we consider the following variational inequality problem:
$(VI)_n$ Find an element $u_n \in K$ such that

$$\begin{aligned} &\int_\Omega A_n\big(x, u_n(x), \nabla u_n(x)\big) \cdot \big(\nabla v(x) - \nabla u_n(x)\big)\, dx\, + \\ &\quad + \int_\Omega a^n\big(x, u_n(x), \nabla u_n(x)\big)\big(v(x) - u_n(x)\big)\, dx\, + \\ &\quad + \int_\Omega l_n\big(x; u_n(x), v(x) - u_n(x)\big)\, dx \geq 0, \forall\, v \in K, \end{aligned}$$

with $A_n = (a_1^n, \ldots, a_N^n)$, $a^n$, and $l_n$ given functions, and $K$ a closed, convex, nonempty subset of the Sobolev space $H^1(\Omega)$.

For a fixed $n \in \mathbb{N}$, we consider the operator

$$\mathcal{A}_n : H^1(\Omega) \to (H^1(\Omega))^*, \qquad \mathcal{A}_n(u) = \mathcal{B}_n(u, u),$$

where

$$\mathcal{B}_n\,(u,v)(w)=\int_\Omega \Big\{\sum_{i=1}^{N} a_i^n(x,u(x),\nabla u(x))\cdot\partial_i w(x)\Big\}dx+\int_\Omega a^n(x,u(x),\nabla u(x))w(x)dx.$$

The operators $\mathcal{A}_n$ are topologically pseudomonotone if satisfies some reasonable conditions (see [8], page 76; [2]).

$(VI)_n$ can be written as

$$\text{find } u_n\in K \text{ such that } f_n(u_n,v)+L_n(u_n,v)\geq 0, \forall\ v\in K,$$

where $f_n(u,v)=\langle\mathcal{A}_n(u),v-u\rangle$ and $L_n(u,v)=\displaystyle\int_\Omega l_n\big(x;u(x),v(x)-u(x)\big)\,dx.$

Finally, we can apply Proposition 2 and Proposition 3 to obtain a result on the stability of $(VI)_n$.

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