# On the functional equation of generalized pseudomediality

ADRIAN PETRESCU

"Petru-Maior" University of Târgu-Mureş, Romania

apetrescu@upm.ro

**Abstract**

Based on $D_{i,j}$ and $D_{i-j}$ conditions (see [1]) we give a direct and straightforward method to solve the functional equation of generalized pseudomediality on quasigroups.

2000 Mathematics Subject Classification: 39B52, 20N05, 20N15.

In this paper we solve the functional equation of generalized pseudomediality on quasigroups

$$\alpha_1(\alpha_2(x,y), \alpha_3(z,u)) = \alpha_4(\alpha_5(\alpha_6(x,z),y)u) \tag{1}$$

where $x, y, z, u$ are taken from an arbitrary set $A$ and $\alpha_i$ are quasigroup operations on $A$.

Sokhats'kyi [3] proved that every quadratic parastrophically uncancelable functional equation for four object variables is parastrophically equivalent to the functional equation of mediality (for this equation see [2]) or the functional equation of pseudomediality. He solved this equation using the functional equation of generalized associativity (for this equation see [2]).

The method we used is an example of the application of our results developed in [1].

Let $\alpha_1, \ldots, \alpha_6$ be a six quasigroup operations on $A$ and forming a solution of the functional equation (1). We define $\alpha : A^4 \to A$ by

$$\alpha(x,y,z,u) = \alpha_1(\alpha_2(x,y), \alpha_3(z,u)) = \alpha_4(\alpha_5(\alpha_6(x,z),y),u) \tag{2}$$

It is obvious that $(A, \alpha)$ is 4-quasigroup and conditions $D_{1,2}$, $D_{3,4}$ and $D_{1-3}$ are fulfilled in $(A, \alpha)$ (see [1]).

**Theorem 1.** *In a 4-loop $(A, \alpha)$ condition $D_{1,2} \& D_{3,4} \& D_{1-3}$ holds iff $\alpha(x,y,z,u) = yxzu$ where $(A, \cdot)$ is a binary group.*

*Proof.* Suppose that in 4-loop $(A, \alpha)$ condition $D_{12} \& D_{3,4} \& D_{1-3}$ holds and let $e$ be a unit in this loop. We define

$$x \cdot y = \alpha(e,e,x,y) \tag{3}$$

Then $(A, \cdot)$ is a binary loop with the unit $e$. From

$$\alpha(e, e, z, u) = \alpha(e, e, e, \alpha(e, e, z, u))$$

by condition $D_{3,4}$ we obtain

$$\alpha(x, y, z, u) = \alpha(x, y, e, zu) \tag{4}$$

Now

$$\begin{aligned}
\alpha(x, y, e, zu) &= \alpha(e, y, zu) \quad \text{(condition } D_{1-3}) \\
&= \alpha(e, y, e, x(zu)) \quad \text{(by (4))} \\
&= \alpha(y, e, e, x(zu)) \quad \text{(condition } D_{1,2}) \\
&= \alpha(e, e, y, x(zu)) \quad \text{(condition } D_{1-3}) \\
&= y(x(zu)). \quad \text{(by (4))}
\end{aligned}$$

Thus

$$\alpha(x, y, z, u) = y(x(zu)) \tag{5}$$

Putting $z = u = e$ in (5) we get

$$\alpha(x, y, e, e) = yx \tag{6}$$

From

$$\alpha(x, y, e, e) = \alpha(\alpha(x, y, e, e), e, e, e)$$

by condition $D_{1,2}$ we have

$$\alpha(x, y, z, u) = \alpha(yx, e, z, u) \tag{7}$$

Now

$$\begin{aligned}
\alpha(yx, e, z, u) &= \alpha(e, yx, z, u) \quad \text{(condition } D_{1,2}) \\
&= \alpha(z, yx, e, u) \quad \text{(condition } D_{1-3}) \\
&= \alpha((yx)z, e, e, u) \quad \text{(by (7))} \\
&= \alpha((yx)z, e, u, e) \quad \text{(condition } D_{3,4}) \\
&= \alpha(e, (yx)z, u, e) \quad \text{(condition } D_{1,2}) \\
&= \alpha(u, (yx)z, e, e) \quad \text{(condition } D_{1-3}) \\
&= ((yx)z)u \quad \text{(by (7))}
\end{aligned}$$

Thus

$$\alpha(x, y, z, u) = ((yx)z)u \tag{8}$$

From (5) and (8) we have

$$y(x(zu)) = (yx)z)u \tag{9}$$

Putting $u = e$ in (9) we obtain

$$y(xz) = (yx)z$$

Therefore $(A, \cdot)$ is a group and $\alpha(x, y, z, u) = yxzu$

The converse is obvious. $\qquad \square$

**Theorem 2.** *The set of all solutions of the functional equation of generalized pseudo-mediality over the set of quasigroup operations on an arbitrary set $A$ is described by the realtions*

$$\alpha_1(x,y) = F_1(x) + F_2(y) \qquad \alpha_2(x,y) = F_1^{-1}(F_3(y) + F_4(x))$$
$$\alpha_3(x,y) = F_2^{-1}(F_5(x) + F_6(y)), \quad \alpha_4(x,y) = F_7(x) + F_6(y) \qquad (10)$$
$$\alpha_5(x,y) = F_7^{-1}(F_3(y) + F_8(x)), \quad \alpha_6(x,y) = F_8^{-1}(F_4(x) + F_5(y))$$

*where $(A, +)$ is an arbitrary group and $F_1, \ldots, F_8$ are arbitrary substitutions of the set $A$.*

*Proof.* Let $\alpha_1, \ldots, \alpha_6$ be six quasigroup operations on $A$ and forming a solution of equation (1). We define $\alpha : A^4 \to A$ by (2). From the above results if follows that $\alpha(x,y,z,u) = T_2(y) + T_1(x) + T_3(z) + T_4(u)$, where $(A, +)$ is a group with zero element $0 = T_1(a_1) = T_2(a_2) = T_3(a_3) = T_4(a_4)$, $T_i$ being translations by $a = (a_1^4) \in A^4$ in $(A, \alpha)$ (for details see [1] and [2]). Putting $z = a_3$ and $u = a_4$ in (2) we get

$$\alpha_1(\alpha_2(x,y), \alpha_3(a_3, a_4)) = \alpha_4(\alpha_5(\alpha_6(x, a_3), y), a_4) = T_2(y) + T_1(x). \qquad (11)$$

The mappings $f(x) = \alpha_1(x, \alpha_3(a_3, a_4)), f_1(x) = \alpha_6(x, a_3)$ and $f_2(x) = \alpha_4(x, a_4)$ are substitutions of the set $A$. From (11) we obtain

$$\alpha_2(x,y) = f^{-1}(T_2(y) + T_1(x))$$

and

$$\alpha_5(x,y) = f_2^{-1}(T_2(y) + T_1(f_1^{-1}(x))).$$

For $x = a_1$ and $y = a_2$ in (2) we have

$$\alpha_1(\alpha_2(a_1, a_2), \alpha_3(z, u)) = \alpha_4(\alpha_5(\alpha_6(a_1, z), a_2), u) = T_3(z) + T_4(u)$$

and thus

$$\alpha_3(z, u) = g^{-1}(T_3(z) + T_4(u))$$

where

$$g(x) = \alpha_1(\alpha_2(a_1, a_2), x)$$

and

$$\alpha_4(z, u) = T_3(g_1^{-1}(z)) + T_4(u)$$

where

$$g_1(x) = \alpha_5(\alpha_6(a_1, x), a_2).$$

Finally, if we put $y = a_2$ and $u = a_4$ in (2) then we have

$$\alpha_1(\alpha_2(x, a_2), \alpha_3(z, a_4)) = \alpha_4(\alpha_5(\alpha_6(x, z), a_2), a_4) = T_1(x) + T_3(z)$$

and thus

$$\alpha_1(x, z) = T_1(h_1^{-1}(x)) + T_3 h_2^{-1}(z)$$

for $h_1(x) = \alpha_2(x, a_2)$ and $h_2(x) = \alpha_3(x, a_4)$,

$$\alpha_6(x, z) = h^{-1}(T_1(x) + T_3(z))$$

where $h(x) = \alpha_4(\alpha_5(x, a_2), a_3)$.

It is easy to prove that

$$f \circ h_1 = T_1, g \circ h_2 = T_3, f_2 \circ g_1 = T_3 \quad \text{and} \quad h \circ f_1 = T_1$$

Taking into account the above results we have

$$\alpha_1(x, z) = f(x) + g(z), \qquad \alpha_2(x, y) = f^{-1}(T_2(y) + T_1(x))$$
$$\alpha_3(z, u) = g^{-1}(T_3(z) + T_4(u)), \quad \alpha_4(z, u) = f_2(z) + T_4(u)$$
$$\alpha_5(x, y) = f_2^{-1}(T_2(y) + h(x)), \quad \alpha_6(x, z) = h^{-1}(T_1(x) + T_3(z)).$$

The converse is clear. $\qquad\square$

# References

[1] Petrescu A., $G - n$-quasigroups, ICTAMI 2007.

[2] Petrescu A., $G - n$-quasigroup and functional equations on quasigroups, IC-TAMI 2007.

[3] Sokats'kyi F.M., On the classification of functional equations on quasigroups, Ukrain. Math. Journal, Vol. 56, No.9 (2004), 1259-1266.