

On the functional equation of generalized pseudomediality

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Abstract

Based on $D_{i,j}$ and D_{i-j} conditions (see [1]) we give a direct and straightforward method to solve the functional equation of generalized pseudomediality on quasigroups.

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In this paper we solve the functional equation of generalized pseudomediality on quasigroups

$$\alpha_1(\alpha_2(x, y), \alpha_3(z, u)) = \alpha_4(\alpha_5(\alpha_6(x, z), y), u) \quad (1)$$

where x, y, z, u are taken from an arbitrary set A and α_i are quasigroup operations on A .

Sokhats'kyi [3] proved that every quadratic parastrophically uncanceled functional equation for four object variables is parastrophically equivalent to the functional equation of mediality (for this equation see [2]) or the functional equation of pseudomediality. He solved this equation using the functional equation of generalized associativity (for this equation see [2]).

The method we used is an example of the application of our results developed in [1].

Let $\alpha_1, \dots, \alpha_6$ be a six quasigroup operations on A and forming a solution of the functional equation (1). We define $\alpha : A^4 \rightarrow A$ by

$$\alpha(x, y, z, u) = \alpha_1(\alpha_2(x, y), \alpha_3(z, u)) = \alpha_4(\alpha_5(\alpha_6(x, z), y), u) \quad (2)$$

It is obvious that (A, α) is 4-quasigroup and conditions $D_{1,2}$, $D_{3,4}$ and D_{1-3} are fulfilled in (A, α) (see [1]).

Theorem 1. *In a 4-loop (A, α) condition $D_{1,2} \& D_{3,4} \& D_{1-3}$ holds iff $\alpha(x, y, z, u) = yxzu$ where (A, \cdot) is a binary group.*

Proof. Suppose that in 4-loop (A, α) condition $D_{1,2} \& D_{3,4} \& D_{1-3}$ holds and let e be a unit in this loop. We define

$$x \cdot y = \alpha(e, e, x, y) \quad (3)$$

Then (A, \cdot) is a binary loop with the unit e . From

$$\alpha(e, e, z, u) = \alpha(e, e, e, \alpha(e, e, z, u))$$

by condition $D_{3,4}$ we obtain

$$\alpha(x, y, z, u) = \alpha(x, y, e, zu) \quad (4)$$

Now

$$\begin{aligned} \alpha(x, y, e, zu) &= \alpha(e, y, zu) \quad (\text{condition } D_{1-3}) \\ &= \alpha(e, y, e, x(zu)) \quad (\text{by (4)}) \\ &= \alpha(y, e, e, x(zu)) \quad (\text{condition } D_{1,2}) \\ &= \alpha(e, e, y, x(zu)) \quad (\text{condition } D_{1-3}) \\ &= y(xzu). \quad (\text{by (4)}) \end{aligned}$$

Thus

$$\alpha(x, y, z, u) = y(xzu) \quad (5)$$

Putting $z = u = e$ in (5) we get

$$\alpha(x, y, e, e) = yx \quad (6)$$

From

$$\alpha(x, y, e, e) = \alpha(\alpha(x, y, e, e), e, e, e)$$

by condition $D_{1,2}$ we have

$$\alpha(x, y, z, u) = \alpha(yx, e, z, u) \quad (7)$$

Now

$$\begin{aligned} \alpha(yx, e, z, u) &= \alpha(e, yx, z, u) \quad (\text{condition } D_{1,2}) \\ &= \alpha(z, yx, e, u) \quad (\text{condition } D_{1-3}) \\ &= \alpha((yx)z, e, e, u) \quad (\text{by (7)}) \\ &= \alpha((yx)z, e, u, e) \quad (\text{condition } D_{3,4}) \\ &= \alpha(e, (yx)z, u, e) \quad (\text{condition } D_{1,2}) \\ &= \alpha(u, (yx)z, e, e) \quad (\text{condition } D_{1-3}) \\ &= ((yx)z)u \quad (\text{by (7)}) \end{aligned}$$

Thus

$$\alpha(x, y, z, u) = ((yx)z)u \quad (8)$$

From (5) and (8) we have

$$y(xzu) = (yx)zu \quad (9)$$

Putting $u = e$ in (9) we obtain

$$y(xz) = (yx)z$$

Therefore (A, \cdot) is a group and $\alpha(x, y, z, u) = yxzu$

The converse is obvious. □

Theorem 2. *The set of all solutions of the functional equation of generalized pseudo-mediality over the set of quasigroup operations on an arbitrary set A is described by the realtions*

$$\begin{aligned} \alpha_1(x, y) &= F_1(x) + F_2(y) & \alpha_2(x, y) &= F_1^{-1}(F_3(y) + F_4(x)) \\ \alpha_3(x, y) &= F_2^{-1}(F_5(x) + F_6(y)), & \alpha_4(x, y) &= F_7(x) + F_6(y) \\ \alpha_5(x, y) &= F_7^{-1}(F_3(y) + F_8(x)), & \alpha_6(x, y) &= F_8^{-1}(F_4(x) + F_5(y)) \end{aligned} \quad (10)$$

where $(A, +)$ is an arbitrary group and F_1, \dots, F_8 are arbitrary substitutions of the set A .

Proof. Let $\alpha_1, \dots, \alpha_6$ be six quasigroup operations on A and forming a solution of equation (1). We define $\alpha : A^4 \rightarrow A$ by (2). From the above results it follows that $\alpha(x, y, z, u) = T_2(y) + T_1(x) + T_3(z) + T_4(u)$, where $(A, +)$ is a group with zero element $0 = T_1(a_1) = T_2(a_2) = T_3(a_3) = T_4(a_4)$, T_i being translations by $a = (a_i^4) \in A^4$ in (A, α) (for details see [1] and [2]). Putting $z = a_3$ and $u = a_4$ in (2) we get

$$\alpha_1(\alpha_2(x, y), \alpha_3(a_3, a_4)) = \alpha_4(\alpha_5(\alpha_6(x, a_3), y), a_4) = T_2(y) + T_1(x). \quad (11)$$

The mappings $f(x) = \alpha_1(x, \alpha_3(a_3, a_4))$, $f_1(x) = \alpha_6(x, a_3)$ and $f_2(x) = \alpha_4(x, a_4)$ are substitutions of the set A . From (11) we obtain

$$\alpha_2(x, y) = f^{-1}(T_2(y) + T_1(x))$$

and

$$\alpha_5(x, y) = f_2^{-1}(T_2(y) + T_1(f_1^{-1}(x))).$$

For $x = a_1$ and $y = a_2$ in (2) we have

$$\alpha_1(\alpha_2(a_1, a_2), \alpha_3(z, u)) = \alpha_4(\alpha_5(\alpha_6(a_1, z), a_2), u) = T_3(z) + T_4(u)$$

and thus

$$\alpha_3(z, u) = g^{-1}(T_3(z) + T_4(u))$$

where

$$g(x) = \alpha_1(\alpha_2(a_1, a_2), x)$$

and

$$\alpha_4(z, u) = T_3(g_1^{-1}(z)) + T_4(u)$$

where

$$g_1(x) = \alpha_5(\alpha_6(a_1, x), a_2).$$

Finally, if we put $y = a_2$ and $u = a_4$ in (2) then we have

$$\alpha_1(\alpha_2(x, a_2), \alpha_3(z, a_4)) = \alpha_4(\alpha_5(\alpha_6(x, z), a_2), a_4) = T_1(x) + T_3(z)$$

and thus

$$\alpha_1(x, z) = T_1(h_1^{-1}(x)) + T_3h_2^{-1}(z)$$

for $h_1(x) = \alpha_2(x, a_2)$ and $h_2(x) = \alpha_3(x, a_4)$,

$$\alpha_6(x, z) = h^{-1}(T_1(x) + T_3(z))$$

where $h(x) = \alpha_4(\alpha_5(x, a_2), a_3)$.

It is easy to prove that

$$f \circ h_1 = T_1, g \circ h_2 = T_3, f_2 \circ g_1 = T_3 \quad \text{and} \quad h \circ f_1 = T_1$$

Taking into account the above results we have

$$\begin{aligned} \alpha_1(x, z) &= f(x) + g(z), & \alpha_2(x, y) &= f^{-1}(T_2(y) + T_1(x)) \\ \alpha_3(z, u) &= g^{-1}(T_3(z) + T_4(u)), & \alpha_4(z, u) &= f_2(z) + T_4(u) \\ \alpha_5(x, y) &= f_2^{-1}(T_2(y) + h(x)), & \alpha_6(x, z) &= h^{-1}(T_1(x) + T_3(z)). \end{aligned}$$

The converse is clear. □

References

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