Applications of Theorem Arens-Michael

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Abstract

We characterize locally, m-convex, complete algebras as being projective limit of projective system of Banach algebras with the aid of Arens-Michael theorem.

1 Introduction

For a normed complex unitary algebra $(A, \|\cdot\|)$ we define the set: $D(A, \|\cdot\|; 1) = \{f \in A' | f(1) = 1 \text{ and } \|f\| = 1\}$. For any $a \in A$ we define numerical range of a the set

$$V(A, \|\cdot\|; a) = \{f(a) | f \in D(A, \|\cdot\|; 1)\},$$

and numerical radius the set:

$$v(A, \|\cdot\|; a) = \sup\{|\lambda| \mid \lambda \in V(A, \|\cdot\|; 1)\}.$$

The set $D(A, \|\cdot\|; 1)$ is a convex subset, weak compact of A' and numerical range $V(A, \|\cdot\|; a)$ is also a compact subset of $\mathbf{C}$, [2].

The properties and applications of numerical ranges on a normed algebra have been largely studied and the main results have been presented by F.F. Bonsall and J.Duncan [2]. The m-convex locally algebras have been thoroughly examined by E.A. Michael in [5].

We observe that for a given m-convex locally algebra A, with unital 1 there exists an increasingly family of submultiplicatively seminormes $\{p_\alpha\}$ on A which generates the topology such that $p_\alpha(1) = 1$ for all α. Given this algebra we denote with $P(A)$ the class of all these family of seminormes on A

and with $(A, \{p_\alpha\})$ the algebra A with the family $\{p_\alpha\}$ fixed by seminormes $\{p_\alpha\} \in P(A)$.

Given $(A, \{p_\alpha\})$ for each α we denote with N_α the null subspace of p_α, through A_α factor subspace $A|_{N_\alpha}$ and with $\|\cdot\|_\alpha$ we denote the norm on A_α, defined by $\|x + N_\alpha\|_\alpha = p_\alpha(x)$. For each α, we consider the linear canonical map $x \mapsto x_\alpha \equiv x + N_\alpha$ from A to A_α. We denote by 1_α the unital element in A_α and it results that $\|1_\alpha\|_\alpha = 1$ for all α. Michael has obtained the significant result that A is isomorph with a subalgebra of the product of normed algebras $(A_\alpha, \|\cdot\|_\alpha)$.

We know that given the unitary algebra A for any $a \in A$, spectrum of a is defined as:

$\sigma(A; a) = \{\lambda \mid a - \lambda \cdot 1 \text{ is non-invertible}\}$. We denote by $\rho(x) := \sup_{\lambda \in \sigma(A,x)} |\lambda|$ the spectral radius of x.

We now that $\rho(x) = \sup_\alpha \lim_{n\to\infty} (p_\alpha(x^n))^{1/n}$.

2 Projective systems. Projective limits

Let be $(A_i)_i \in I$ a family of algebras with I increasingly family, i.e for all $i, j \in I \Rightarrow \exists k \in I,$ cu $i \le k, j \le k$. Let be a family $\{f_{ij}\}_{i,j\in I}$ of morphism of algebras given by:

1. $f_{ij} : A_j \to A_i$ for all $i, j \in I$ with $i \le j$
2. $f_{ii} = id_{A_i}, i \in I$
3. $i, j, k \in I$, cu $i \le j \le k \Rightarrow f_{ik} = f_{ij} \circ f_{jk}$

The family $\{(A_i, f_{ij})\}_{i,j\in I}$ is call *projective system algebras*. We consider cartesian product $F = \prod\limits_{i\in I} A_i$ and the following subset of F:

$A = \{x \in (x_i)_{i\in I} \in F | x_i = f_{ij}(x_j), \text{ dac@a } i \le j \in I\}$.

Theorem 1. *A is a subalgebra of F.*

Proof. We show that A is a subalgebra of F. Let $x, y \in A$, it follows that $x_i + y_i = f_{ij}(x_j + y_j) = f_{ij}(x_j) + f_{ij}(y_j)$. Hence $x + y \in A$. Analogous we show that $\alpha x \in A,\ \forall \alpha \in \mathbf{C},\ \forall x \in A$.

Let $x, y \in A,\ xy = (x_i y_i)_{i\in I},\ f_{ij}(x_j y_j) = f_{ij}(x_j) f_{ij}(y_j) = x_i y_i,$
$\forall i, j \in I$ with $i \le j$. Hence $xy \in A$. $\square$

If each algebra $A_i,\ i \in I$ have unital element, 1_i and morphisms f_{ij}, with $i \le j,\ i, j \in I$ keeps unital, then unital of $F,\ 1 = (1_i)_{i\in I}$ we find in subalgebra A of F.

The algebra A defined above is call projectiv limit algebra of projective system of algebras (A_i, f_{ij}) and to denote trough $A = \varprojlim(A_i, f_{ij})$ or $A = \varprojlim A_i$. On the other hand doing a projective system of algebra to define a family $(f_i)_{i\in I}$ of morphism of algebras trough $f_i = \pi_i|A : A \to A_i$ for all $i \in I$ i.e. considering restrictions to of A projects π_i of F on A_i.

The applications F_i can not to be surjectives.

From the definition of A and definition of applications f_i, it follows that $f_i = f_{ij} \circ f_j$ for any $i, j \in I$ with $i \leq j$.

$f_{ij}(f_j(x)) = f_{ij}(x_j) = x_i = f_i(x)$.

Definition 1. *Let be (A_i, f_{ij}) a projective system of algebras, where A_i for any $i \in I$ is a topological algebra and f_{ij} ,$i, j \in I, i \leq j$, continuous morphism of algebras.*

The system (A_i, f_{ij}) is called a projective system of topological algebras.

The projective limits A is endowed with initial topology defined by family $(f_i)_{i\in I}$. This topological algebra is named projective limit of topological algebras.

Proof. A projective limit of locally m-convex algebras is an algebra. □

Lemma 1. *Let $A = \varprojlim(A_i, f_{ij})$ a projective limit of topological algebras. Then the family $\{f_i^{-1}(U_i)|U_i \in \mathcal{V}_i, i \in I\}$, where $\mathcal{V}_i$ represents a fundamentally system by neighborhoods of 0 from algebra $A_i, i \in I$ is a fundamentally system by neighborhoods of 0 for A.*

Proof. Let $f_i = \pi_i|_A :\to A_i$. We have that

$$\bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_{i_j}) \text{ implies } \bigcap_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}), \tag{1}$$

where U_{i_j} belong of a fundamentally system of neighborhoods of 0, from A_{i_j}, is a fundamentally system of neighborhoods of 0 on $\prod_{i\in I} A_i$. Since I is a increasingly set $i \in I$, there exists $i_j \leq i, j = 1, ..., n$ such that $f_{i_j} = f_{i_j i} \circ f_i$. Hence

$$\bigcap_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}) = \bigcap_{j=1}^{n} f_i^{-1}\left(f_{i_j i}^{-1}\left(U_{i_j}\right)\right) = \tag{2}$$

$$= f_i^{-1}\left(\bigcap_{j=1}^{n} f_{i_j i}^{-1}\left(U_{i_j}\right)\right) = f_i^{-1}(V_i), \tag{3}$$

whre $V_i = \bigcap_{j=1}^{n} f_{i_j}^{-1}(U_{i_j})$ is a neighborhood in A_I. There exists $U_i \in \mathcal{V}_i$, such that $U_i \subseteq V_i$ which shows assertions of enunciation. $\square$

Lemma 2. *Any projective limit of tolopogical algebras $A = \varprojlim A_i$ is a closed subalgebra of topological algebra cartesian product $F = \prod_{i \in I} A_i$. Particularly, A is complete if each $A_i, i \in I$ from topological algebras is complete.*

Proof. Let $x \in \overline{A}$. Then there exists $(x^\alpha)_{\alpha \in J}$, $x^\alpha \in A$, such that $x^\alpha \to x$ if and only if $x^\alpha - x \to 0$ if and only if $x_i^\alpha \to x_i$, for all $i \in I$. From $x^a \in A$ it follows that

$$f_{ij}(x_j^\alpha) = x_i^\alpha,\ (\forall) i \leq j,\ i, j \in I. \tag{4}$$

From the continuity of f_{ij} it follows that

$$f_{ij}(x_j) = x_i,\ (\forall) i \leq j,\ i, j \in I.$$

Hence $x \in A$. If each $A_i, i \in I$ is complete then the cartesian product A is a complete algebra and how A is a closed space it follows that is complete. $\square$

Lemma 3. *Let $A = \varprojlim(A_i, f_{ij})$ a projective limit of topological algebra and B is a subalgebra of A. Then*

$$\overline{B} = \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right) = \varprojlim \overline{f_i(B)},$$

where $f_i = \pi_i|_A : A \to A_i, (\forall) i \in I$. Particularly, if B is closed, then

$$B = \varprojlim f_i(B) = \varprojlim \overline{f_i(B)}.$$

Proof. We observe that the family $\left\{\left(f_i(B), f_{ij}|_{f_j(B)}\right)\right\}_{i \in I}$ defines a projective system of topological algebras which it follows that from continuity of f_{ij} with $i \leq j$ and from $f_i = f_{ij} \circ f_j$ for $i \leq j, i, j \in I$.

On the other hand:

$$f_{ij}\left(\overline{f_j(B)}\right) \subseteq \overline{f_{ij}(f_j(B))} = \overline{f_i(B)}.$$

For $i \leq j$ we obtain the family $\left\{\left(\overline{f_i(B)}, f_{ij}|_{\overline{f_j(B)}}\right)\right\}_{i \in I}$ defines also a projective system of topological algebras.

Immediately from definition it follows that

$$\varprojlim \overline{f_i(B)} = \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right).$$

We show that $\overline{B} = \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right)$.

Let $x \in \overline{B}$. It follows that there exists a generalized sequence of B, $(x^\alpha)_{\alpha \in J}$, such that $x^\alpha \to x$.

From the continuity of f_i, fo rall $i \in I$, it results
$f_i(x^\alpha) \to f_i(x), (\forall)x \in I \Rightarrow f_i(x) \in \overline{f_i(B)} \Rightarrow$
$\Rightarrow x \in f_i^{-1}\left(\overline{f_i(B)}\right), (\forall)i \in I$, hence

$$x \in \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right).$$

Conversely, let $x \in \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right)$. It follows that $x \in f_i^{-1}\left(\overline{f_i(B)}\right)$, $(\forall)i \in I$. It follows that $f_i(x) \in \overline{f_i(B)}$, for all $i \in I$, hence

$$x_i \in \overline{f_i(B)}, (\forall)i \in I,$$

it follows that for all $U_i \in \mathcal{V}(x_i)$ we have $U_i \cap f_i(B) \neq \emptyset$ it follows that $f_i^{-1}(U_i) \cap B \neq \emptyset$, hence $x \in \overline{B}$ since $f_i^{-1}(U_i)$ is a fundamental system by neighborhoods of x. On the other hand we have

$$B \subseteq \bigcap_{i \in I} f_i^{-1}\left(f_i(B)\right) \subseteq \bigcap_{i \in I} f_i^{-1}\left(\overline{f_i(B)}\right).$$

Then $B \subseteq \lim\limits_{\longleftarrow} f_i(B) \subseteq \lim\limits_{\longleftarrow} \overline{f_i(B)} = \overline{B}$, which prove last part of lemma. □

2.1 Arens-Michael Theorem

Theorem 2. *(Arens-Michael)*

Let be A a locally, m-convex algebra and $\mathcal{V} = (U_i)_{i \in I}$ a fundamental system of neighborhoods of 0 equilibrates, convex, multiplicative and absorbent. We denote with $\tilde{A}$ expanding of A and let be A_i (respective $\tilde{A}_i$), $i \in I$ normed algebras (Banach) suitable to fundamental system by neighborhoods $\mathcal{V}$ defined above. Then

$$A \hookrightarrow \lim_{\leftarrow} A_i \hookrightarrow \lim_{\leftarrow} \tilde{A}_i = \tilde{A}_i,$$

where $\hookrightarrow$ is injective, topologic morphism of algebras. Particularly, for each locally, m-convex, complete algebra A, with fundamental system by neighborhoods $\mathcal{V}$, we obtain

$$A = \lim_{\leftarrow} A_i = \lim_{\leftarrow} \tilde{A}_i,$$

where " = " signifies topological isomorphism of algebras (surjective morphism).

Proof. Let be $(p_i)_{i\in I}$ a increasingly family of seminorms associated of $\mathcal{V}$. Then $A_i = A/_{N_i} = \{x + N_i | x \in A\} = \{x_i\}, i \in I$ where $N_i = p_i^{-1}(0)$, $x_i = x + N_i, i \in I$. Let be $a \in A$, $x \in N_i$. Then $p_i(ax) \leq p_i(a) \cdot p_i(x)$. Since $p_i(x) = 0$ it follows that $p_i(ax) = 0$. Therefore, $ax \in N_i$. It follows that N_i is an ideal. Then $A/_{N_i}$ is an algebra.

We define $\|x_i\|_i = p_i(x)$ which is obviously norm on A_i. Let $f_{ij} : A_j \to A_i$, $\forall i, j \in I,\ j \geq i$, given:

$$f_{ij}(x + N_j) = x + N_i.$$

Obviously, f_{ij} is a morphism of algebras. Let:
(1) $f_{ij}((x_1 + N_j)(x_2 + N_j)) = f_{ik}(x_1x_2 + N_j) = x_1x_2 + N_i =$
$= (x_1 + N_i)(x_2 + N_i) = f_{ij}(x_1 + N_j)f_{ij}(x_2 + N_j)$
We have that $\|f_{ij}(x + N_j)\|_i = p_i(x) \leq p_j(x) = \|x + N_i\|_j$; it follows that f_{ij} continuous. We have:
(2) $f_{ii}(x + N_i) = x + N_i$.
Let $i \leq j \leq k$. Then
(3) $f_{ik} = f_{ij} \circ f_{jk}(x + N_k) = f_{ij}(x + N_j) = x + N_i = f_{ik}(x + N_k)$,
hence $f_{ik} = f_{ij} \circ f_{jk}$, for all $i \leq j \leq k$, $i, j, k \in I$.
Therefore, from relations (1), (2), (3) the family $\{(A_i, f_{ij})\}$ form a projective system of normed algebras.

Since $(A_i)_{i\in I}$ (respective $(\tilde{A}_i)_{i\in I}$) is a family of normed algebras (respective Banach) and form a projective system of normed algebras it follows that from above propositions that exists $\lim\limits_{\leftarrow} A_i$ (respectively $\lim\limits_{\leftarrow} \tilde{A}_i$) and is a locally m-convex algebra.

Considering locally m-convex algebras $\lim\limits_{\leftarrow} A_i$ and $\lim\limits_{\leftarrow} \tilde{A}_i$, we define the following application:

$$\varphi : A \to \lim_{\leftarrow} A_i$$

through

$$\varphi(x) = (\varphi_i(x))_{i\in I} = (x_i)_{i\in I}$$

where $\varphi_i(x) = x + N_i = x + \mathrm{Ker}(p_i) = x_i$, $i \in I$.

To show that φ is well defined to observe that $\varphi_i = f_{ij} \circ \varphi_j$

Indeed, $f_{ij}(\varphi_j(x)) = \varphi_i(x)$. It follows that $\varphi(x) \in \lim\limits_{\leftarrow} A_i$. It is obviously that φ is a morphism of algebras. From $\varphi(u) = 0$ it follows that $(\varphi_i(x))_{i\in I} = 0$ hence $\varphi_i(x) = 0$, $i \in I$. Hence $p_i(x) = 0, i \in I$. Therefore, $x = 0$ (A algebra separate) and so φ is a injective morphism, hence isomorphic of algebras.

We prove now that φ is topological isomorphism.

We have $f_i \circ \varphi_i = \varphi_i$, for all $i \in I$.

Indeed, $(f_i \circ \varphi)(x) = f_i(\varphi(x)) = f_i(\varphi_i(x)_{i\in I}) = \varphi_i(x)$. Since f_i and φ_i are continue functions $i \in I$ it follows that φ is continuous.

The inverse functions $\varphi^{-1} : \lim\limits_{\leftarrow} A_i \to A$ is also continuous. Indeed if $U_i \in \mathcal{V}$ then

$$V = \left(\prod_{j\in I} V_j\right) \cap \varphi(A),$$

with $U_i = \varphi_i\left(\frac{1}{2}U_i\right)$ and $V_j = A_j$, for any $j \in V, j \neq i$ is a neighborhoods of 0 in $\varphi(A)$ with properties that $V \subseteq \varphi(U_i)$, which means $\varphi^{-1}(V) \subseteq U_i$ and hence φ^{-1} is continuous.

We verify that $V \subseteq \varphi(U_i)$. Let $y \in V \Rightarrow y \in \prod\limits_{j\in J} V_j$ and $y \in \varphi(A)$, it follows that there exists $x \in A$ with $y = \varphi(x)$ and $\varphi_i(x) = \varphi_i(\frac{1}{2}z)$, with $z \in U_i$.

Hence $p_i(x) = p_i\left(\frac{1}{2}z\right) = \frac{1}{2}p_i(z) \leq \frac{1}{2} < 1 \Rightarrow$
$\Rightarrow x \in U_i \Rightarrow y = \varphi(x)$ with $x \in U_i \Rightarrow y = \varphi(x) \in \varphi(U_i) \Rightarrow$
$\Rightarrow V \subseteq \varphi(U_i) \Rightarrow \varphi^{-1}(V) \subseteq U_i$.

Therefore, φ is surjective, topologic morphism of algebras of A to into $\lim\limits_{\leftarrow} A_i$.

We have canonical injections $\theta_i : A_i \to \tilde{A}_i, i \in I$ which are algebraical and topological isomorphisms and commutes with the maps f_{ij} and with their extensions $\tilde{f}_{ij} : \tilde{a} \to \tilde{A}_i$, with $i \leq j$.

We obtain:

$$\theta : \lim_{\leftarrow} A_i \to \lim_{\leftarrow} \tilde{A}_i$$

defined through $\theta(x) = (\theta_i(x_i)), i \in I$.

$$\tilde{f}_{ij}(\theta_j(x_j)) = \tilde{f}_{ij}(x_j) = f_{ij}(x_j) = x_i = \theta_i(x_i)$$

Isomorphism θ is topologic too, which it follows immediately from the definitions of algebraical topologies.

On the other hand, since

$$f_i \circ \varphi = \varphi_i, (\forall)i \in I, \quad \varphi_i : A \to A_i = A/_{\mathrm{Ker}_{\varphi_i}}$$

we have:

$$f_i(\varphi(A)) = \varphi_i(A) = A_i, i \in I$$

and then from above lema and from conclusions for φ we obtain

$$A \subseteq \overline{A} = \overline{\varphi(A)} = \lim_{\leftarrow} \overline{f_i(\varphi(A))} = \lim_{\leftarrow} A_i = \lim_{\leftarrow} \tilde{A}_i.$$

Since the last space is complete it follows that from above lemma $\tilde{A} = \overline{A} = \lim\limits_{\leftarrow} \overline{A}_i = \lim\limits_{\leftarrow} \tilde{A}$ where equality represents isomorphisms of studied algebras above.

Therefore from θ is a topological isomorphism the proof is end. $\square$

2.2 Applications of Arens-Michael theorem

Theorem 3. *Let be A a locally, m-convex and complete algebra and $A = \lim\limits_{\leftarrow} A_i$.*

1) *The algebra A has unital element if and only if $\tilde{A}_i$ has unital element for all $i \in I$.*

2) *An element $x \in A$ is invertible if and only if $\varphi_i(x)$ is invertible in $\tilde{A}_i$ for any $i \in I$.*

Proof. We suppose $1 = (1_i) \in \prod\limits_{i \in I} \tilde{A}_i$, with 1_i unital element in $\tilde{A}_i$ for $i \in I$.

Since $A_i = \varphi_i(A)$, if $x_i = \varphi_i(x) \in A_i$, then

$$x_i \tilde{f}_{ij}(1_j) = \varphi_i(x)\tilde{f}_{ij}(1_j) = \tilde{f}_{ij}(\varphi_j(x))\tilde{f}_{ij}(1_j) = \tilde{f}_{ij}(\varphi_j(x)1_j) =$$

$$= \tilde{f}_{ij}(\varphi_j(x)) = \varphi_i(x) = x_i$$

for any $i \leq j$ and similarly for left multiplication with $\tilde{f}_{ij}(1_j)$, hence $\tilde{f}_{ij}(1_j)$ is a unital for A_i, hence also for $\tilde{A}_i = \overline{A}_i$. Then it follows that $\tilde{f}_{ij}(1_j) = 1_i$ for any $i \leq j$ from I, hence $1 = (1_i)_{i \in I} \in \lim\limits_{\leftarrow} \tilde{A}_i = A$.

Verify that 1 is unital element of A.

We prove (2). If $x = (x_i)_{i \in I} \in A = \lim\limits_{\leftarrow} A_i$, how x_i is invertible of $\tilde{A}_i$ for any $i \in I$, there exist $y = (y_i)_{i \in I} \in \prod\limits_{i \in I} \tilde{A}_i$ such that:

$$x_i \cdot y_i = y_i \cdot x_i = 1_i,$$

where from (1) we know that $(1_i)_{i \in I} = 1$ is a unital element of A.

Now for $i \leq j$ from I, we obtain:
$x_i \cdot \tilde{f}_{ij}(y_j) = \tilde{f}_{ij}(x_j) \cdot \tilde{f}_{ij}(y_j) = \tilde{f}_{ij}(x_j \cdot y_j) =$
$= \tilde{f}_{ij}(1_j) = 1_i = \tilde{f}_{ij}(y_j) \cdot x_i$, which means that $\tilde{f}_{ij}(y_j)$ is inverse of x_i in $\tilde{a}_i$ and then we deduce $\tilde{f}_{ij}(y_j) = y_i$, for $i \leq j$ from I.
Hence $y = (y_i)_{i \in I} \in \lim\limits_{\leftarrow} \tilde{A}_i$. From above we deduce that y is inverse of x in A. □

Corollary 1. *Let be A a locally, m-convex and complete algebra and $x \in A$. Then:*

$$\sigma(A, x) = \bigcup_{i \in I} \sigma(A, x_i),$$

$$\rho(x) = \sup_{i \in I} \rho(x_i) = \sup_{i \in I} \lim_{n \to \infty} (p_i(x^n))^{\frac{1}{n}}.$$

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