

Property of Regular Excessive Measures

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Abstract

We show that a measure $\xi \in \text{Exc}_{\mathcal{U}}$ will be regular if and only if exists $\mu \in \mathcal{M}_+(E)$ such that $\xi = \mu \circ U$, where μ is a σ -finite measure on E which does not charge any \mathcal{B} -measurable semipolar set.

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Introduction

We consider a proper sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin measurable space (E, \mathcal{B}) such that the set $\mathcal{E}_{\mathcal{U}}$ of all \mathcal{B} -measurable, \mathcal{U} -excessive functions on E which are \mathcal{U} -a.e finite is min-stable, contains the positive constant functions and generates the σ -algebra \mathcal{B} .

We suppose that the set E is semisaturated (see [3]) with respect to \mathcal{U} i.e. any \mathcal{U} -excessive measure dominated by a potential is also potential. We denote by $\text{Exc}_{\mathcal{U}}$ the set of all \mathcal{U} -excessive measures on E (see [11], [12]). The specific order \preceq on $\text{Exc}_{\mathcal{U}}$ is defined as follows: if $\xi, \xi' \in \text{Exc}_{\mathcal{U}}$ then $\xi \preceq \xi'$ if and only if there exists $\eta \in \text{Exc}_{\mathcal{U}}$ such that $\xi + \eta = \xi'$.

For every σ -finite measure μ on (E, \mathcal{B}) such that exists $\xi \in \text{Exc}_{\mathcal{U}}$ with $\mu \leq \xi$, we put $R(\mu) = \wedge \{\xi \in \text{Exc}_{\mathcal{U}} \mid \mu \leq \xi\}$.

1 Semipolar sets

Throughout the paper $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be a proper sub-Markovian resolvent on (E, \mathcal{B}) as in Introduction. If $A \subset E$ and $s \in \mathcal{E}_\mathcal{U}$ then réduite of s on A is the function $R^A s$ on E defined by $R^A s := \inf\{t \in \mathcal{E}_\mathcal{U} \mid t \geq s \text{ on } A\}$. If $A \in \mathcal{B}$ and $s \in \mathcal{E}_\mathcal{U}$ then $R^A s$ is universally measurable (see [5]) and we denote by $B^A s$ its \mathcal{U} -excessive regularization. (The fine topology is the topology on E generated by $\mathcal{E}_\mathcal{U}$). The function $B^A s$ is called the baleage of s on A .

For every $A \in \mathcal{B}^u$ we denote by A^* the set given by

$$A^* = \{x \in A \mid \liminf_n nU_n(1_A)(x) = 1\},$$

where 1_A is the characteristic function of A . Clearly $A^* \in \mathcal{B}^u$ and $A^* \in \mathcal{B}$ provided that $A \in \mathcal{B}$.

Theorem 1. *For every $A \in \mathcal{B}^u$ the following assertions hold:*

1. *If A is finely open then $A = A^*$.*
2. *The set $A \setminus A^*$ is \mathcal{U} -negligible and for each $s \in \mathcal{E}_\mathcal{U}$ the function $R^{A^*} s$ is \mathcal{U} -excessive and there exists a sequence $(f_n)_n$ in $bp\mathcal{B}^u$ such that*

$$Uf_n \uparrow R^{A^*} s \quad \text{and} \quad f_n = 0 \quad \text{on} \quad E \setminus A^* \tag{1}$$

3. *If $A \in \mathcal{B}$ and $s \in \mathcal{E}_\mathcal{U}$ then $R^{A^*} s \in \mathcal{E}_\mathcal{U}$ and there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ such that (1) holds.*

For the proof see Theorem 1.3.8 [6].

It is known that if $A \in \mathcal{B}$ then one has $B^A s = R^A s$ on $E \setminus A$ (see [5]).

If $\xi \in \text{Exc}_\mathcal{U}$ a subset A of E is called ξ -polar provided that there exists a \mathcal{U} -excessive function s on E such that $s = +\infty$ on A and $s < \infty$ ξ - a.e. If μ is a σ -finite measure on E such that $\mu \circ U \in \text{Exc}_\mathcal{U}$ then we say μ -polar instead of $\mu \circ U$ polar. A set A is called polar if it is ξ -polar for every $\xi \in \text{Exc}_\mathcal{U}$.

A subset M of E is called nearly \mathcal{B} -measurable provided that for every finite measure μ on (E, \mathcal{B}) there exists a \mathcal{B} -measurable set $M_0 \subset M$ such that the set $M \setminus M_0$ is μ -polar and μ -negligible.

We denote by \mathcal{B}^n the family of all nearly \mathcal{B} -measurable subsets of E (see [6]). Obviously \mathcal{B}^n is a σ -algebra on E and $\mathcal{B} \subset \mathcal{B}^n \subset \mathcal{B}^u$ (the set of all universally \mathcal{B} -measurable subsets of E).

Recall that a set $A \in \mathcal{B}$ is *thin* at a point $x \in E$ if there exists $s \in \mathcal{E}_{\mathcal{U}}$ such that $B^A s(x) < s(x)$. A subset M of E is called thin at x if there exists $A \in \mathcal{B}$ such that $M \subset A$, which is thin at x . The set M is called *totally thin* if it is thin at any point of E . For any subset M of E the set $b(M) := \{x \in E \mid M \text{ is not thin at } x\}$ is usually called the *base* of M . It is a fine closed set and $b(M) = b(\overline{M}^f) \subset \overline{M}^f$, where \overline{M}^f denotes the fine closure of M . If M is nearly \mathcal{B} -measurable and $p := Uf_0$ is bounded with f_0 \mathcal{B} -measurable, $0 < f_0 \leq 1$ then we have

$$M \text{ is thin at } x \Leftrightarrow B^M p(x) < p(x)$$

$$b(M) = [B^M p = p].$$

A subset M of E is called *basic* (respective *subbasic*) if $b(M) = M$ (resp. $M \subset b(M)$). If M is subbasic then \overline{M}^f is basic.

A subset of E is termed *semipolar* if it is a countable union of totally thin sets.

Remark. Since $R^{A^*} s \in \mathcal{E}_{\mathcal{U}}$ for any $A \in \mathcal{B}^u$ (see [5]) it follows that $R^{A^*} s = \mathcal{B}^{A^*} s$ for all $s \in \mathcal{E}_{\mathcal{U}}$. Hence $B^{A^*} Uf_0 = Uf_0$ on A^* , where $0 < f_0 \leq 1$, f_0 \mathcal{B} -measurable such that Uf_0 is bounded. Therefore A^* is a subbasic set.

2 The regular excessive measures

We recall that an element $\xi \in \text{Exc}_{\mathcal{U}}$ is called *regular* if for any increasing sequence $(\xi_n)_{n \in \mathbb{N}} \subset \text{Exc}_{\mathcal{U}}$ such that $\bigvee_{n \in \mathbb{N}} \xi_n = \xi$ we have $\bigwedge_{n \in \mathbb{N}} R(\xi - \xi_n) = 0$ (see [3]).

Note (see [5], [6]) that a \mathcal{U} -excessive measure $\xi = \mu \circ U$ is regular iff μ is a σ -finite measure on E which does not charge any \mathcal{B} -measurable semipolar set.

For the proof see Theorem 3.4.5 [6].

Theorem 2. *Let E be a semisaturated set with respect to \mathcal{U} . Then a measure $\xi \in \text{Exc}_{\mathcal{U}}$ is regular if and only if exists $\mu \in \mathcal{M}_+(E)$ such that $\xi = \mu \circ U$, where μ is a σ -finite measure on E which does not charge any \mathcal{B} -measurable semipolar set.*

Proof. Let $(\mu_n)_n$ a sequence of $\mathcal{M}_+(E)$ such that $\mu_n \circ U \uparrow \xi$. Because ξ is regular we get $R(\xi - \mu_n \circ U) \downarrow 0$. Putting $\xi_n = R(\xi - \mu_n \circ U)$ we get $\xi_n \preceq \xi$ and excessive measure $\eta_n := \xi - \xi_n$ is such that $\eta_n \leq \mu_n \circ U$ i.e. there exists $\nu_n \in \mathcal{M}_+(E)$ with $\eta_n = \nu_n \circ U$ and $\eta_n \preceq \xi$. From $\xi_n \downarrow 0$ we have that $\eta_n \uparrow \xi$. We denote by $\xi' = \bigvee_n \eta_n$. Because $\eta_n + \xi_n = \xi$ we get $\bigvee_{k=1}^n \eta_k + \bigwedge_{k=1}^n \xi_k = \xi$. But from $\xi_n \downarrow 0$ we have that $\bigvee_n \eta_n = \xi$. Therefore $\xi' = \xi$. Putting $\bigvee_{k \leq n} \eta_k = \eta'_n$ it results that there exists $\nu'_n \in \mathcal{M}_+(E)$ such that $\eta'_n = \nu'_n \circ U$. Because $\bigvee_n \eta_n = \bigvee_n \eta'_n$ and putting $\mu = \bigvee_n \nu'_n$ we get $\xi = \mu \circ U$. Therefore ξ is regular if and only if $\mu \circ U$ is regular i.e. iff μ does not charge any \mathcal{B} -measurable semipolar set. \square

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