

# GENERATED MOSCO CONVERGENCE FOR SETS

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## Abstract

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. We emphasize some formalisms in normed spaces that generate convergent sequences of sets. A different approach for a kind of restriction sets that proves Mosco convergence is given.

**Key words:** Parametric problems, Equilibrium problems, Variational inequalities, Mosco convergence, Topological pseudomonotonicity.

There are numerous problems in nonlinear analysis, like scalar and/or vector equilibrium problems, scalar and/or vector variational inequality problems, where parametric domains occur (see [2, 5, 6]).

Let  $X$  be a normed space.

Let  $A, A_n, n \in \mathbb{N}$  be nonempty sets in  $X$ . We shall use the following notations:

$$\liminf A_n = \{x \in X \mid \exists(x_n), x_n \in A_n, \forall n \in \mathbb{N}, x_n \rightarrow x\};$$

$$\limsup A_n = \{x \in X \mid \exists(n_k), \exists(x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \rightarrow x\};$$

$$w - \limsup A_n = \{x \in X \mid \exists(n_k), \exists(x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \xrightarrow{w} x\},$$

where  $w$  denotes the weak convergence in  $X$ .

## Definition 1.

(PK) *The sequence  $(A_n)_{n \in \mathbb{N}}$  is said to converges in the Painlevé-Kuratowski sense to  $A$  and notes  $A_n \xrightarrow{PK} A$  if*

$$\limsup A_n \subseteq A \subseteq \liminf A_n;$$

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(M) The sequence  $(A_n)_{n \in \mathbb{N}}$  is said to converges in the Mosco sense to  $A$  and notes  $A_n \xrightarrow{M} A$  if

$$w - \limsup A_n \subseteq A \subseteq \liminf A_n.$$

It is clear that if  $A_n \xrightarrow{M} A$  then  $A_n \xrightarrow{PK} A$ .

It is easy to provide examples in  $\mathbb{R}$ . Let  $n \geq 1$ ,  $A_n = \{-1/n, n\}$  for  $n$  odd, and  $A_n = \{1/n\}$  for  $n$  even. Then  $A_n \xrightarrow{PK} A$ , where  $A = \{0\}$ .

The parametrization concept can be done for the discrete mode, i.e. the parameter is  $n \in \mathbb{N}$  or a continuous mode when the parameter is considered in a topological (or metric) space. It is the last case when the parameter is the time  $t$  in an interval  $I \subseteq \mathbb{R}$  (see [3]). A slight generalization for sets convergence has been used in this case. Let  $(X, \sigma)$  be a Hausdorff topological spaces, let  $P$  (the set of parameters) be also a Hausdorff topological space and let  $p_0 \in P$  be fixed. Along with the topology  $\sigma$ , we also consider a stronger topology  $\tau$  on  $X$ . If  $X$  is a normed space and  $\sigma = \tau =$  norm topology,  $D_p \xrightarrow{M} D_{p_0}$  amounts to saying that the sets  $D_p$  converge to  $D_{p_0}$  in the Painlevé-Kuratowski sense as  $p \rightarrow p_0$ . If  $X$  is a normed space and  $\sigma$  is chosen as the weak topology and  $\tau$  as the norm topology, then obviously,  $D_p \xrightarrow{M} D_{p_0}$  implies  $D_p \xrightarrow{PK} D_{p_0}$  as  $p \rightarrow p_0$ . It is useful the following form.

**Definition 2.** Let  $D_p$  be subsets of  $X$  for all  $p \in P$ . The sets  $D_p$  converge to  $D_{p_0}$  (and write  $D_p \xrightarrow{M} D_{p_0}$ ) as  $p \rightarrow p_0$  if:

- (a) for every net  $(a_{p_i})_{i \in I}$  with  $a_{p_i} \in D_{p_i}$ ,  $p_i \rightarrow p_0$  and  $a_{p_i} \xrightarrow{\sigma} a$  imply  $a \in D_{p_0}$ ;
- (b) for every  $a \in D_{p_0}$ , there exist  $a_p \in D_p$  such that  $a_p \xrightarrow{\tau} a$  as  $p \rightarrow p_0$ .

In [1] is described a concrete situation in order to generate Mosco convergence. For a reflexive Banach space  $X$ , a set-valued mapping  $D : X \rightarrow 2^X$  is defined by

$$D(x) = D_0 + d(x),$$

where  $D_0$  is a closed convex nonempty subset of  $X$  and  $d : X \rightarrow X$  is a compact map (i.e. a weakly-strongly continuous map).

**Proposition 1.** ([1], Proposition 1) For any sequence  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \xrightarrow{w} x$  in  $X$ , then  $D(x_n) \xrightarrow{M} D(x)$ .

A partial result from [1], Theorem 1 can be deduced from our result Theorem 1 in [4].

Are there other formalisms that generate Mosco convergence for sets? Certainly, among them see Lemma 1.4, 1.6 in [6].

This type of convergence was introduced by Umberto Mosco in [6]. His examples were the following:

**Example 1.** Let  $X = l^2 = \{y = (y_k)_{k \in \mathbb{N}} \mid \sum_{k=1}^{\infty} |y_k|^2 < +\infty\}$ . Let  $B$  the unit ball in  $X$  with respect to the usual norm  $\|y\| = (\sum_{k=1}^{\infty} |y_k|^2)^{1/2}$ . Denote by  $S_0 = (2B) \cap \{y = (y_k)_{k \in \mathbb{N}} \mid 0 \leq y_k \leq 1, \forall k \in \mathbb{N}\}$  and  $S_n = (2B) \cap \{y = (y_k)_{k \in \mathbb{N}} \mid 0 \leq y_k \leq 1 + k/n, \forall k \in \mathbb{N}\}$ . One has  $S_n \xrightarrow{M} S_0$ .

Let  $E = \{e_k \mid k \in \mathbb{N}\}$  be the canonical basis. Denote by  $C_0 = \text{cl conv}\{e_1, e_2, \dots, e_k, \dots\}$  and  $C_n = \text{cl conv}\{(1+1/n)e_1, (1+2/n)e_2, \dots, (1+k/n)e_k, \dots\}$ . One has  $C_n \xrightarrow{M} C_0$ .

Now, conform Proposition 1.2.1 in [2], for  $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}$ , one has

- $\limsup(A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n$
- $\liminf(A_n \cap B_n) \subseteq \liminf A_n \cap \liminf B_n$ .

The question rises: if  $A_n \xrightarrow{PK} A$  and  $B_n \xrightarrow{PK} B$  when  $A_n \cap B_n \xrightarrow{PK} A \cap B$ ? The same: if  $A_n \xrightarrow{M} A$  and  $B_n \xrightarrow{M} B$  when  $A_n \cap B_n \xrightarrow{M} A \cap B$ ?

There is an answer given in [7], Proposition 27 along with the constrained condition  $X = \mathbb{R}_+(A - B)$ .

In [3] the following kind of sets occurs:

$$K(t) = \{x(t) \in \mathbb{R}^m : \lambda(t) \leq x(t) \leq \mu(t), M(t) \cdot x(t) = b(t)\},$$

where  $\lambda, \mu \in C([0, T], \mathbb{R}^m)$  with  $\lambda < \mu$  and  $b \in C([0, T], \mathbb{R}^l_+)$  are vector-functions and  $M \in C([0, T], \mathbb{R}^{l \times m}_+)$  is a matrix-function.

**Proposition 2.** ([3]) *Let  $(t_n)_{n \in \mathbb{N}} \subset [0, T]$  be a sequence such that  $t_n \rightarrow t$ . Then*

$$K(t_n) \xrightarrow{PK} K(t), \text{ as } n \rightarrow \infty.$$

We shall consider a more simple case but a different approach. Define the closed, convex sets

$$A(t) = \{x(t) \in \mathbb{R}^m \mid \lambda(t) \leq x(t)\}.$$

Since the product sets enjoys the following properties:

- $\limsup \prod_{k=1}^m A_n^k \subseteq \prod_{k=1}^m \limsup A_n^k$ ;
- $\liminf \prod_{k=1}^m A_n^k = \prod_{k=1}^m \liminf A_n^k$ ,

we can conclude that  $A(t_n) \xrightarrow{PK} A(t)$ , as  $n \rightarrow \infty$  if this is proved for  $m = 1$ .

Let  $x(t) \in A(t)$ . Define

$$x(t_n) := x(t) - \lambda(t) + \lambda(t_n).$$

We have  $x(t_n) \in A(t_n)$  and by the continuity of  $\lambda$  we get  $\lim_{n \rightarrow \infty} x(t_n) = x(t)$ , that is  $A(t) \subseteq \liminf A(t_n)$ . Now, let  $(x(t_{n_k}))_{k \in \mathbb{N}}$  be a subsequence with  $x(t_{n_k}) \in A(t_{n_k})$ ,  $k \in \mathbb{N}$  such that  $x(t_{n_k}) \rightarrow x(t)$ . Passing to the limit for  $k \rightarrow \infty$  in

$$\lambda(t_{n_k}) \leq x(t_{n_k})$$

using again the continuity of  $\lambda$  we get  $x(t) \in A(t)$ , therefore  $\limsup A(t_n) \subseteq A(t)$ . Analogously  $B(t) = \{x(t) \in \mathbb{R} \mid x(t) \leq \mu(t)\}$ . In this case, since  $\lambda < \mu$  we have  $\mathbb{R} = \mathbb{R}_+(A(t) - B(t))$ .

The above result is no longer true provided  $\lambda \in L^2[0, T]$ . Anyway, following [1] if one defines

$$K(\lambda) = \{x \in H^1[0, T] \mid \int_{[0, T]} (x - \lambda)(t) dt \geq 0\}$$

it was proved that  $K(\lambda_n) \xrightarrow{M} K(\lambda)$  as  $\lambda_n \xrightarrow{w} \lambda$  in  $L^2[0, T]$ .

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. That result can be viewed as a formalism for the static case. In our future work we shall be concerned on the dynamic case, precisely where the parametric domains and functions are depending on time.

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