

Direct Decompositions of Quasigroups and Homotopies

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Abstract

In this paper we investigate direct decompositions in the category **QGR** whose objects are n -quasigroups and morphisms are quasigroup homotopies.

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1 Introduction

In the theory of n -quasigroups ($n \geq 2$) the role played by homotopies is as important as that played by homomorphisms. But, in many applications of n -quasigroups, isotopies and homotopies are more important than isomorphisms and homomorphisms. So, the study of homotopic properties of algebraic constructions become important.

Direct products give a means of creating n -quasigroups of huge order (applications in cryptography) than what we start with. A direct product of a family of n -quasigroups is completely determined by its factors.

The aim of the present paper is to present direct decomposition of n -quasigroups in the category **QGR** whose morphisms are quasigroup homotopies.

The second section records absolute and weak permutability of equivalence relations. Quasigroup homotopies kernels are presented in section 3. Section 4 examines direct decompositions.

To simplify the notation, we will omit the prefix n in n -quasigroup.

2 Permutability of equivalence relations

We recall two generalizations of the permutability of equivalence relations.

Let $\mathcal{S} = \{q_j \mid j \in J\}$ be a family of equivalence relations on a set A .

\mathcal{S} is called **absolutely permutable** [4] if it satisfies the following condition: for any family $\{a_j \mid j \in J\}$ if $a_j \equiv a_k(\vee q_j)$ for all $j, k \in J$ there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$.

\mathcal{S} is called **weakly permutable** [1] if it satisfies the following condition: for any family $\{a_j \mid j \in J\}$ if $a_j \equiv a_k(q_j \vee q_k)$ for all $j, k \in J$ there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$.

The concept of weak permutability is weaker than that of absolute permutability. Some useful results are:

Theorem 1. *The following are equivalent:*

- (i) \mathcal{S} is absolutely permutable;
- (ii) \mathcal{S} is weakly permutable and for any $q_j, q_k \in \mathcal{S}$, if $q_j \neq q_k$, then $q_j \vee q_k = \vee\{q_j \mid j \in J\}$.

Theorem 2. *If \mathcal{S} is absolutely permutable then $q_j \circ \bar{q}_j = \vee\{q_j \mid j \in J\}$, where $\bar{q}_j = \wedge\{q_k \mid k \in J, k \neq j\}$, for all $j \in J$. If J is finite the converse is true.*

We will simplify several proofs in section 4 using the following result.

Theorem 3. *Let A and J be two sets. There exists a bijective map $f : A \rightarrow \prod B_j$ the cartesian product of the family $\{B_j \mid j \in J\}$ of sets if and only if there exists a family $\mathcal{S} = \{q_j \mid j \in J\}$ of equivalence relations on A such that:*

- (i) $\wedge q_j = \Delta_A$;
- (ii) $\vee q_j = A^2$;
- (iii) \mathcal{S} is absolutely permutable.

Proof. Suppose $f : A \rightarrow \prod B_j$. Let be $\mathcal{S} = \{q_j \mid j \in J\}$ where $q_j = \ker(p_j f)$, p_j being the j -th projection. We have $\Delta_A = \ker(f) = \wedge \ker(p_j f) = \wedge q_j$. Let $a, a' \in A$. Choose $j, k \in J$, and consider an element $b \in \prod B_j$ such that $p_j(b) = p_j f(a)$ and $p_k(b) = p_k f(a')$. The map f being surjective there exists $a^* \in A$ such that $b = f(a^*)$. Then $p_j f(a^*) = p_j(b) = p_j f(a)$ and $p_k f(a^*) = p_k(b) = p_k f(a')$ imply $a \equiv a^*(q_j)$ and $a^* \equiv a'(q_k)$, i.e., $a \equiv a'(q_j \circ q_k)$. In consequence $\vee q_j = A^2$. Let now $\{a_j \mid j \in J\}$ be a family of elements in A . Consider $f(a^*) = b \in \prod B_j$ such that $p_j(b) = p_j f(a_j)$. Then $p_j f(a^*) = p_j(b) = p_j f(a_j)$, i.e., $a^* \equiv a_j(q_j)$, $j \in J$.

Conversely, let be $\mathcal{S} = \{q_j \mid j \in J\}$ such that conditions (i)-(iii) are satisfied. Consider the map $f : A \rightarrow \prod A/q_j$ such that $p_j f = \text{nat}_{q_j} : A \rightarrow A/q_j$. The map f is injective: $\ker(f) = \wedge(p_j f) = \wedge q_j = \Delta_A$. For an element $b \in \prod A/q_j$ let be $a_j \in A$ such that $p_j f(a_j) = p_j(b)$. Taking into account (ii) and (iii) there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$. Hence $p_j f(a) = p_j f(a_j) = p_j(b)$, $j \in J$ imply $f(a) = b$. \square

3 Normal congruent families of equivalence relations

In this section, we collect some definitions and results that will be used later. For a more detailed exposition, the reader is referred to [2] and [3].

Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be quasigroups. A **homotopy** $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is an ordered system of maps $\varphi = [f_1, \dots, f_n; f]$ from the set A to the set B such that

$$f\alpha(x_1, \dots, x_n) = \beta(f_1(x_1), \dots, f_n(x_n)) \quad (1)$$

for all $x_1, \dots, x_n \in A$.

The map f_i , $i \in \mathbb{N}_n = \{1, 2, \dots, n\}$ is known as the i -th component of φ and f -the principal component. The equality and composition of homotopies are defined componentwise.

The category **QGR** has the class of all quasigroups as its object class and its morphisms are quasigroup homotopies. Isomorphisms in **QGR** are called isotopies. They are just the homotopies having each component bijective.

The **kernel of homotopy** φ is $\ker(\varphi) = [\ker(f_1), \dots, \ker(f_n); \ker(f)]$.

A **normal congruent family of equivalences** θ on a quasigroup $\mathcal{A} = (A, \alpha)$ is an ordered system of equivalence relations on the set A , $\theta = [q_1, \dots, q_n; q]$, such that for all $a = (a_1, \dots, a_n) \in A^n$.

$$T_i^2(q_i) = q, \text{ for all } i \in \mathbb{N}_n \quad (2)$$

where $T_i : A \rightarrow A$, $T_i(x) = \alpha(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ is the i -th elementary transposition by a .

The kernel of homotopy $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a normal congruent family of equivalences on \mathcal{A} .

We show that the converse is also true.

Let $\theta = [q_1, \dots, q_n, q]$ be a normal congruent family of equivalences on $\mathcal{A} = (A, \alpha)$. For any $a = (a_1, \dots, a_n) \in A$ $T_i^* : A/q_i \rightarrow A/q$, $T_i^*(q_i(x)) = q(T_i(x))$ is bijective for each $i \in \mathbb{N}_n$ ($q_i(x), q(x)$ – the blocks of x).

We define an n -ary operation $\bar{\alpha}$ on A/q by

$$\bar{\alpha}(q(x_1), \dots, q(x_n)) = q(\alpha(T_1^{-1}(x_1), \dots, T_n^{-1}(x_n))) \quad (3)$$

Then $(A/q, \bar{\alpha})$ is a loop having $e = \alpha(a_1, \dots, a_n)$ as a unit.

It is easy to see that $\varphi = [f_1, \dots, f_n; f] : \mathcal{A} \rightarrow (A/q, \bar{\alpha})$ defined by $f_i(x) = q(T_i(x))$, $i \in \mathbb{N}_n$ and $f(x) = q(x)$ is a homotopy and $\ker(\varphi) = \theta$.

The operation $\bar{\alpha}$ depends on a . For an another element $b = (b_1, \dots, b_n) \in A^n$ we obtain an another loop $(A/q, \beta)$. They are principal isotopic. So, the notation $\mathcal{A}/\theta = (A/q, \alpha)$ is consistent. We call \mathcal{A}/θ a **quotient quasigroup of \mathcal{A} by θ** .

For an n -quasigroup \mathcal{A} , let $NCF(\mathcal{A})$ denote the set of normal congruent families of equivalences on \mathcal{A} . Define an order relation \leq on $NCF(\mathcal{A})$ by setting

$$\theta_1 = [q_{11}, \dots, q_{n1}; q_1] \leq \theta_2 = [q_{12}, \dots, q_{n2}; q_2]$$

iff $q_{i1} \subseteq q_{i2}$, $i \in \mathbb{N}_n$ and $q_1 \subseteq q_2$.

If $\mathcal{S} = \{\theta_j = [q_{1j}, \dots, q_{nj}; q_j] \mid j \in J\}$ is a family of normal congruent families of equivalences on \mathcal{A} then

$$\wedge \theta_j = [\wedge q_{1j}, \dots, \wedge q_{nj}; \wedge q_j]$$

and

$$\vee \theta_j = [\vee q_{1j}, \dots, \vee q_{nj}; \vee q_j]$$

are again normal congruent families of equivalences on \mathcal{A} . Thus $NCF(\mathcal{A})$ forms a complete lattice under \leq .

Now let be $\mathcal{S} = \{\theta_j \mid j \in J\} \subseteq NCF(\mathcal{A})$. Then $\mathcal{S}_i = \{q_{ij} \mid j \in J\}$, $i \in \mathbb{N}_n$ and $\mathcal{S}_{n+1} = \{q_j \mid j \in J\}$ are families of equivalence relations on A called **components of \mathcal{S}** . By (2), if one component of \mathcal{S} is absolutely (weakly) permutable then all components are absolutely (weakly) permutable.

We call \mathcal{S} **absolutely (weakly) permutable** if all its components are absolutely (weakly) permutable.

4 Direct decompositions in QGR

We present the homotopic properties of direct products of quasigroups.

Let $\{\mathcal{A}_j = (A_j, \alpha_j) \mid j \in J\}$ be a family of quasigroups. The direct product of this family is the quasigroup $\prod \mathcal{A}_j = (\prod A_j, \alpha)$ whose underlying set is the cartesian product $\prod A_j$ and operation α is defined coordinatewise. The projections $p_j : \prod \mathcal{A}_j \rightarrow \mathcal{A}_j$, $j \in J$, $p((a_j)_{j \in J}) = a_j$, $j \in J$ are quasigroup homomorphisms.

Theorem 4. *The category **QGR** has products.*

Proof. Let $(\prod \mathcal{A}_j, \{p_j \mid j \in J\})$ is a product in **QGR**. Indeed, let be $\varphi_j = [f_{1j}, \dots, f_{nj}; f_j] : \mathcal{B} \rightarrow \mathcal{A}_j$, $j \in J$. Consider the maps $f_i, f : \mathcal{B} \rightarrow \prod \mathcal{A}_j$ defined by $p_j f_i = f_{ij}$, $i \in \mathbb{N}_n$ and $p_j f = f_j$, $j \in J$. It is easy to show that $\varphi = [f_1, \dots, f_n; f] : \mathcal{B} \rightarrow \prod \mathcal{A}_j$ is the unique homotopy with $p_j \varphi = \varphi_j$, $j \in J$. \square

Let \mathcal{A} be a quasigroup and let $\{\mathcal{A}_j \mid j \in J\}$ be a family of quasigroups.

Definition 1. *A decomposition of \mathcal{A} as a **direct product of $\{\mathcal{A}_j \mid j \in J\}$ is a QGR-isomorphism (quasigroup isotopy) $\varphi : \mathcal{A} \rightarrow \prod \mathcal{A}_j$. The decomposition is called **proper** if none of the homotopies $p_j \varphi$ is a QGR-monomorphism (a homotopy with all component injective). \mathcal{A} is called **direct indecomposable** if it admits no proper direct decomposition.***

Theorem 5. *\mathcal{A} has a proper direct decomposition iff there exists a family $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:*

- (i) $\wedge \theta_j = \Delta_A$;
- (ii) $\vee \theta_j = A^2$;
- (iii) \mathcal{S} is absolutely permutable.

Proof. Let $\varphi = [f_1, \dots, f_n] : \mathcal{A} \rightarrow \prod \mathcal{A}_j$ be a proper direct decomposition. Put $\theta_j = \ker(p_j \varphi)$, $j \in J$. Taking into account Theorem 3 it is easy to show that $\mathcal{S} = \{\theta_j \mid j \in J\}$ satisfies conditions (i) – (iii).

Conversely, suppose that $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ satisfies conditions (i) – (iii). It is easy to see that all its components satisfy conditions (i) – (iii). Consider the direct product $\prod \mathcal{A}/\theta_j$ and the homotopy

$$\varphi = [f_1, \dots, f_n; f] : \mathcal{A} \rightarrow \prod \mathcal{A}/\theta_j$$

defined by $p_j \varphi = \varphi_j$ where $\varphi_j : \mathcal{A} \rightarrow \mathcal{A}/\theta_j$ are the canonical homotopies defined in previous section.

By Theorem 3 it follows that φ is a proper direct decomposition of \mathcal{A} . \square

By Theorem 5 and Theorem 1 we get

Theorem 6. \mathcal{A} has a proper direct decomposition iff there exists a family $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:

- (i) $\wedge \theta_j = \Delta_A$;
- (ii) $\theta_j \vee \theta_k = A^2$, for any $\theta_j, \theta_k \in \mathcal{S}$, $\theta_j \neq \theta_k$;
- (iii) \mathcal{S} is weakly permutable.

By Theorem 5 and Theorem 2 we get

Theorem 7. (Chinese remainder theorem). \mathcal{A} has a finite proper direct decomposition iff there exists a finite family $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:

- (i) $\wedge \theta_j = \Delta_A$;
- (ii) $\theta_j \circ \bar{\theta}_j = A^2$, $j \in J$.

The following theorem is useful to characterize direct indecomposable quasigroups.

Theorem 8. \mathcal{A} has a proper direct decomposition iff there exists $\theta_1, \theta_2 \in NCF(\mathcal{A})$ such that:

- (i) $\theta_1, \theta_2 > \Delta_A$, $\theta_1 \neq \theta_2$;
- (ii) $\theta_1 \wedge \theta_2 = \Delta_A$;
- (iii) $\theta_1 \circ \theta_2 = A^2$.

Proof. Suppose that \mathcal{A} has a proper direct decomposition. Let be $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\}$ as in Theorem 5. There exists $\theta_i \in \mathcal{S}$ such that $\theta_i < A^2$. Then $\bar{\theta}_i > \Delta_A$, and $\theta_i \wedge \bar{\theta}_i = \Delta_A$. By Theorem 2, $\theta_i \circ \bar{\theta}_i = A^2$.

The converse follows by Theorem 7. □

Corollary 1. \mathcal{A} is direct indecomposable iff there is no pair $\theta_1, \theta_2 \in NCF(\mathcal{A})$, $\theta_1 \neq \theta_2$ with

- (i) $\theta_1, \theta_2 > \Delta_A$;
- (ii) $\theta_1 \wedge \theta_2 = \Delta_A$;
- (iii) $\theta_1 \circ \theta_2 = A^2$.

A quasigroup direct indecomposable in the subcategory **Qgr** (whose morphisms are quasigroup homomorphisms) of **QGR** can be proper decomposable in **QGR**.

Example. Let $\mathcal{A} = (A, \cdot)$ be the binary quasigroup.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 3 | 4 | 6 | 7 | 1 | 2 | 5 | 8 |
| 2 | 4 | 3 | 7 | 6 | 2 | 1 | 8 | 5 |
| 3 | 7 | 6 | 4 | 3 | 8 | 5 | 2 | 1 |
| 4 | 6 | 7 | 3 | 4 | 5 | 8 | 1 | 2 |
| 5 | 1 | 2 | 5 | 8 | 3 | 4 | 6 | 7 |
| 6 | 2 | 1 | 8 | 5 | 4 | 3 | 7 | 6 |
| 7 | 8 | 5 | 2 | 1 | 7 | 6 | 4 | 3 |
| 8 | 5 | 8 | 1 | 2 | 6 | 7 | 3 | 4 |

It is easy to see that only Δ_A and A^2 are normal congruences on \mathcal{A} . Hence \mathcal{A} is direct indecomposable in Qgr . \mathcal{A} is direct indecomposable in \mathbf{Qgr} , but \mathcal{A} has a proper direct decomposition in \mathbf{QGR} : $\theta = [q_1, q_2; q]$ defined by

$$\begin{aligned} A/q_1 &= \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}\} \\ A/q_2 &= \{\{1, 4\}, \{2, 3\}, \{5, 8\}, \{6, 7\}\} \\ A/q &= \{\{1, 8\}, \{2, 5\}, \{3, 7\}, \{4, 6\}\} \end{aligned}$$

and $\theta' = [q'_1, q'_2; q']$ defined by

$$\begin{aligned} A/q'_1 &= A/q'_2 = \{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\} \\ A/q' &= \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\} \end{aligned}$$

verify conditions of Theorem 8.

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