Direct Decompositions of Quasigroups and Homotopies

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Abstract

In this paper we investigate direct decompositions in the category **QGR** whose objects are n-quasigroups and morphisms are quasigroup homotopies.

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1 Introduction

In the theory of n-quasigroups ($n \geq 2$) the role played by homotopies is as important as that played by homomorphisms. But, in many applications of n-quasigroups, isotopies and homotopies are more important than isomorphisms and homomorphisms. So, the study of homotopic properties of algebraic constructions become important.

Direct products give a means of creating n-quasigroups of huge order (applications in cryptography) than what we start with. A direct product of a family of n-quasigroups is completely determined by its factors.

The aim of the present paper is to present direct decomposition of n-quasigroups in the category **QGR** whose morphisms are quasigroup homotopies.

The second section records absolute and weak permutability of equivalence relations. Quasigroup homotopies kernels are presented in section 3. Section 4 examines direct decompositions.

To simplify the notation, we will omit the prefix n in n-quasigroup.

2 Permutability of equivalence relations

We recall two generalizations of the permutability of equivalence relations.

Let $\mathcal{S} = \{q_j \mid j \in J\}$ be a family of equivalence relations on a set A.

$\mathcal{S}$ is called **absolutely permutable** [4] if it satisfies the following condition: for any family $\{a_j \mid j \in J\}$ if $a_j \equiv a_k(\vee q_j)$ for all $j, k \in J$ there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$.

$\mathcal{S}$ is called **weakly permutable** [1] if it satisfies the following condition: for any family $\{a_j \mid j \in J\}$ if $a_j \equiv a_k(q_j \vee q_k)$ for all $j, k \in J$ there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$.

The concept of weak permutability is weaker than that of absolute permutability. Some useful results are:

Theorem 1. *The following are equivalent:*

(i) $\mathcal{S}$ *is absolutely permutable;*

(ii) $\mathcal{S}$ *is weakly permutable and for any* $q_j, q_k \in \mathcal{S}$, *if* $q_j \neq q_k$, *then* $q_j \vee q_k = \vee\{q_j \mid j \in J\}$.

Theorem 2. *If* $\mathcal{S}$ *is absolutely permutable then* $q_j \circ \overline{q}_j = \vee\{q_j \mid j \in J\}$, *where* $\overline{q}_j = \wedge\{q_k \mid k \in J, k \neq j\}$, *for all* $j \in J$. *If* J *is finite the converse is true.*

We will simplify several proofs in section 4 using the following result.

Theorem 3. *Let* A *and* J *be two sets. There exists a bijective map* $f : A \to \sqcap B_j$ *the cartesian product of the family* $\{B_j \mid j \in J\}$ *of sets if and only if there exists a family* $\mathcal{S} = \{q_j \mid j \in J\}$ *of equivalence relations on* A *such that:*

(i) $\wedge q_j = \triangle_A$;

(ii) $\vee q_j = A^2$;

(iii) $\mathcal{S}$ *is absolutely permutable.*

Proof. Suppose $f : A \to \sqcap B_j$. Let be $\mathcal{S} = \{q_j \mid j \in J\}$ where $q_j = \ker(p_j f)$, p_j being the j-th projection. We have $\triangle_A = \ker(f) = \wedge \ker(p_j f) = \wedge q_j$. Let $a, a' \in A$. Choose $j, k \in J$, and consider an element $b \in \sqcap B_j$ such that $p_j(b) = p_j f(a)$ and $p_k(b) = p_k f(a')$. The map f being surjective there exists $a^* \in A$ such that $b = f(a^*)$. Then $p_j f(a^*) = p_j(b) = p_j f(a)$ and $p_k f(a^*) = p_k(b) = p_k f(a')$ imply $a \equiv a^*(q_j)$ and $a^* \equiv a'(q_k)$, i.e., $a \equiv a'(q_j \circ q_k)$. In consequence $\vee q_j = A^2$. Let now $\{a_j \mid j \in J\}$ be a family of elements in A. Consider $f(a^*) = b \in \sqcap B_j$ such that $p_j(b) = p_j f(a_j)$. Then $p_j f(a^*) = p_j(b) = p_j f(a_j)$, i.e., $a^* \equiv a_j(q_j)$, $j \in J$.

Conversely, let be $\mathcal{S} = \{q_j \mid j \in J\}$ such that conditions (i)-(iii) are satisfied. Consider the map $f : A \to \sqcap A/q_j$ such that $p_j f = \mathrm{nat} q_j : A \to A/q_j$. The map f is injective: $\ker(f) = \wedge(p_j f) = \wedge q_j = \triangle_A$. For an element $b \in \sqcap A/q_j$ let be $a_j \in A$ such that $p_j f(a_j) = p_j(b)$. Taking into account (ii) and (iii) there exists $a \in A$ such that $a \equiv a_j(q_j)$ for all $j \in J$. Hence $p_j f(a) = p_j f(a_j) = p_j(b)$, $j \in J$ imply $f(a) = b$. $\square$

3 Normal congruent families of equivalence relations

In this section, we collect some definitions and results that will be used later. For a more detailed exposition, the reader is referred to [2] and [3].

Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be quasigroups. A **homotopy** $\varphi : \mathcal{A} \to \mathcal{B}$ is an ordered system of maps $\varphi = [f_1, \ldots, f_n; f]$ from the set A to the set B such that

$$f\alpha(x_1, \ldots, x_n) = \beta(f_1(x_1), \ldots, f_n(x_n)) \tag{1}$$

for all $x_1, \ldots, x_n \in A$.

The map f_i, $i \in \mathbb{N}_n = \{1, 2, \ldots, n\}$ is known as the i-th component of φ and f-the principal component. The equality and composition of homotopies are defined componentwise.

The category **QGR** has the class of all quasigroups as its object class and its morphisms are quasigroup homotopies. Isomorphisms in **QGR** are called isotopies. They are just the homotopies having each component bijective.

The **kernel of homotopy** φ is $\ker(\varphi) = [\ker(f_1), \ldots, \ker(f_n); \ker(f)]$.

A **normal congruent family of equivalences** θ on a quasigroup $\mathcal{A} = (A, \alpha)$ is an ordered system of equivalence relations on the set A, $\theta = [q_1, \ldots, q_n; q]$, such that for all $a = (a_1, \ldots, a_n) \in A^n$.

$$T_i^2(q_i) = q, \text{ for all } i \in \mathbb{N}_n \tag{2}$$

where $T_i : A \to A$, $T_i(x) = \alpha(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is the i-th elementary translation by a.

The kernel of homotopy $\varphi : \mathcal{A} \to \mathcal{B}$ is a normal congruent family of equivalences on $\mathcal{A}$.

We show that the converse is also true.

Let $\theta = [q_1, \ldots, q_n, q]$ be a normal congruent family of equivalences on $\mathcal{A} = (A, \alpha)$. For any $a = (a_1, \ldots, a_n) \in A$ $T_i^* : A/q_i \to A/q$, $T_i^*(q_i(x)) = q(T_i(x))$ is bijective for each $i \in \mathbb{N}_n$ ($q_i(x), q(x)$ – the blocks of x).

We define an n-ary operation $\overline{\alpha}$ on A/q by

$$\overline{\alpha}(q(x_1), \ldots, q(x_n)) = q(\alpha(T_1^{-1}(x_1), \ldots, T_n^{-1}(x_n)) \tag{3}$$

Then $(A/q, \overline{\alpha})$ is a loop having $e = \alpha(a_1, \ldots, a_n)$ as a unit.

It is easy to see that $\varphi = [f_1, \ldots, f_n; f] : \mathcal{A} \to (A/q, \overline{\alpha})$ defined by $f_i(x) = q(T_i(x))$, $i \in \mathbb{N}_n$ and $f(x) = q(x)$ is a homotopy and $\ker(\varphi) = \theta$.

The operation $\overline{\alpha}$ depends on a. For an another element $b = (b_1, \ldots, b_n) \in A^n$ we obtain an another loop $(A/q, \beta)$. They are principal isotopic. So, the notation $\mathcal{A}/\theta = (A/q, \alpha)$ is consistent. We call $\mathcal{A}/\theta$ a **quotient quasigroup of** $\mathcal{A}$ by θ.

For an n-quasigroup $\mathcal{A}$, let $NCF(\mathcal{A})$ denote the set of normal congruent families of equivalences on $\mathcal{A}$. Define an order relation $\leq$ on $NCF(\mathcal{A})$ by setting

$$\theta_1 = [q_{11}, \ldots, q_{n1}; q_1] \leq \theta_2 = [q_{12}, \ldots, q_{n2}; q_2]$$

iff $q_{i1} \subseteq q_{i2}$, $i \in \mathbb{N}_n$ and $q_1 \subseteq q_2$.

If $\mathcal{S} = \{\theta_j = [q_{ij}, \ldots, q_{nj}; q_j] \mid j \in J\}$ is a family of normal congruent families of equivalences on $\mathcal{A}$ then

$$\wedge\theta_j = [\wedge q_{1j}, \ldots, \wedge q_{nj}; \wedge q_j]$$

and

$$\vee\theta_j = [\vee q_{1j}, \ldots, \vee q_{nj}; \vee q_j]$$

are again normal congruent families of equivalences on $\mathcal{A}$. Thus $NCF(\mathcal{A})$ forms a complete lattice under $\leq$.

Now let be $\mathcal{S} = \{\theta_j \mid j \in J\} \subseteq NCF(\mathcal{A})$. Then $\mathcal{S}_i = \{q_{ij} | j \in J\}$, $i \in \mathbb{N}_n$ and $\mathcal{S}_{n+1} = \{q_j \mid j \in J\}$ are families of equivalence relations on A called **components of** $\mathcal{S}$. By (2), if one component of $\mathcal{S}$ is absolutely (weakly) permutable then all components are absolutely (weakly) permutable.

We call $\mathcal{S}$ **absolutely (weakly) permutable** if all its components are absolutely (weakly) permutable.

4 Direct decompositions in QGR

We present the homotopic properties of direct products of quasigroups.

Let $\{\mathcal{A}_j = (A_j, \alpha_j) \mid j \in J\}$ be a family of quasigroups. The direct product of this family is the quasigroup $\sqcap \mathcal{A}_j = (\sqcap A_j, \alpha)$ whose underlying set is the cartesian product $\sqcap A_j$ and operation α is defined coordinatewise. The projections $p_j : \sqcap \mathcal{A}_j \to \mathcal{A}_j$, $j \in J$, $p((a_j)_{j\in J}) = a_j$, $j \in J$ are quasigroup homomorphisms.

Theorem 4. *The category **QGR** has products.*

Proof. Let $(\sqcap \mathcal{A}_j, \{p_j \mid j \in J\})$ is a product in **QGR**. Indeed, let be $\varphi_j = [f_{1j}, \ldots, f_{nj}; f_j] : \mathcal{B} \to \mathcal{A}_j$, $j \in J$. Consider the maps $f_i, f : B \to \sqcap A_j$ defined by $p_j f_i = f_{ij}$, $i \in \mathbb{N}_n$ and $p_j f = f_j$, $j \in J$. It is easy to show that $\varphi = [f_1, \ldots, f_n; f] : \mathcal{B} \to \sqcap \mathcal{A}_j$ is the unique homotopy with $p_j f = \varphi_j$, $j \in J$. □

Let $\mathcal{A}$ be a quasigroup and let $\{\mathcal{A}_j \mid j \in J\}$ be a family of quasigroups.

Definition 1. *A decomposition of $\mathcal{A}$ as a **direct product of** $\{\mathcal{A}_j \mid j \in J\}$ is a **QGR**-isomorphism (quasigroup isotopy) $\varphi : \mathcal{A} \to \sqcap \mathcal{A}_j$. The decomposition is called **proper** if none of the homotopies $p_j\varphi$ is a **QGR**- monomorphism (a homotopy with all component injective). $\mathcal{A}$ is called **direct indecomposable** if it admits no proper direct decomposition.*

Theorem 5. *$\mathcal{A}$ has a proper direct decomposition iff there exists a family $\mathcal{S} = \{\theta_j > \triangle_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:*
(i) $\wedge \theta_j = \triangle_A$;
(ii) $\vee \theta_j = A^2$;
(iii) $\mathcal{S}$ is absolutely permutable.

Proof. Let $\varphi = [f_1, \ldots, f_n] : \mathcal{A} \to \sqcap \mathcal{A}_j$ be a proper direct decomposition. Put $\theta_j = \ker(p_j f)$, $j \in J$. Taking into account Theorem 3 it is easy to show that $\mathcal{S} = \{\theta_j \mid j \in J\}$ satisfies conditions (i) – (iii).

Conversely, suppose that $\mathcal{S} = \{\theta_j > \triangle_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ satisfies conditions (i) – (iii). It is easy to see that all its components satisfy conditions (i) – (iii). Consider the direct product $\sqcap \mathcal{A}/\theta_j$ and the homotopy

$$\varphi = [f_1, \ldots, f_n; f] : \mathcal{A} \to \sqcap \mathcal{A}/\theta_j$$

defined by $p_j\varphi = \varphi_j$ where $\varphi_j : \mathcal{A} \to \mathcal{A}/\theta_j$ are the canonical homotopies defined in previous section.

By Theorem 3 it follows that φ is a proper direct decomposition of $\mathcal{A}$. □

By Theorem 5 and Theorem 1 we get

Theorem 6. *$\mathcal{A}$ has a proper direct decomposition iff there exists a family $\mathcal{S} = \{\theta_j > \triangle_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:*
(i) $\wedge\theta_j = \triangle_A$;
(ii) $\theta_j \vee \theta_k = A^2$, *for any* θ_j, $\theta_k \in \mathcal{S}$, $\theta_j \neq \theta_k$;
(iii) $\mathcal{S}$ *is weakly permutable.*

By Theorem 5 and Theorem 2 we get

Theorem 7. *(Chinese remainder theorem). $\mathcal{A}$ has a finite proper direct decomposition iff there exists a finite family $\mathcal{S} = \{\theta_j > \triangle_A \mid j \in J\} \subseteq NCF(\mathcal{A})$ such that:*
(i) $\wedge\theta_j = \triangle_A$;
(ii) $\theta_j \circ \overline{\theta}_j = A^2$, $j \in J$.

The following theorem is useful to characterize direct indecomposable quasigroups.

Theorem 8. *$\mathcal{A}$ has a proper direct decomposition iff there exists $\theta_1, \theta_2 \in NCF(\mathcal{A})$ such that:*
(i) $\theta_1, \theta_2 > \triangle_A$, $\theta_1 \neq \theta_2$;
(ii) $\theta_1 \wedge \theta_2 = \triangle_A$;
(iii) $\theta_1 \circ \theta_2 = A^2$.

Proof. Suppose that $\mathcal{A}$ has a proper direct decomposition. Let be $\mathcal{S} = \{\theta_j > \triangle_A \mid j \in J\}$ as in Theorem 5. There exists $\theta_i \in \mathcal{S}$ such that $\theta_i < A^2$. Then $\overline{\theta}_i > \triangle_A$, and $\theta_i \wedge \overline{\theta}_i = \triangle_A$. By Theorem 2, $\theta_i \circ \overline{\theta}_i = A^2$.

The converse follows by Theorem 7. □

Corollary 1. *$\mathcal{A}$ is direct indecomposable iff there is no pair $\theta_1, \theta_2 \in NCF(\mathcal{A})$, $\theta_1 \neq \theta_2$ with*
(i) $\theta_1, \theta_2 > \triangle_A$;
(ii) $\theta_1 \wedge \theta_2 = \triangle_A$;
(iii) $\theta_1 \circ \theta_2 = A^2$.

A quasigroup direct indecomposable in the subcategory **Qgr** (whose morphisms are quasigroup homomorphisms) of **QGR** can be proper decomposable in **QGR**.

Example. Let $\mathcal{A} = (A, \cdot)$ be the binary quasigroup.

	1	2	3	4	5	6	7	8
1	3	4	6	7	1	2	5	8
2	4	3	7	6	2	1	8	5
3	7	6	4	3	8	5	2	1
4	6	7	3	4	5	8	1	2
5	1	2	5	8	3	4	6	7
6	2	1	8	5	4	3	7	6
7	8	5	2	1	7	6	4	3
8	5	8	1	2	6	7	3	4

It is easy to see that only $\triangle_A$ and A^2 are normal congruences on $\mathcal{A}$. Hence $\mathcal{A}$ is direct indecomposable in Qgr. $\mathcal{A}$ is direct indecomposable in **Qgr**, but $\mathcal{A}$ has a proper direct decomposition in **QGR**: $\theta = [q_1, q_2; q]$ defined by

$$
\begin{aligned}
A/q_1 &= \{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\}\\
A/q_2 &= \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}\\
A/q &= \{\{1,8\},\{2,5\},\{3,7\},\{4,6\}\}
\end{aligned}
$$

and $\theta' = [q_1', q_2'; q']$ defined by

$$
\begin{aligned}
A/q_1' = A/q_2' &= \{\{1,2,5,6\},\{3,4,7,8\}\}\\
A/q' &= \{\{1,2,3,4\},\{5,6,7,8\}\}
\end{aligned}
$$

verify conditions of Theorem 8.

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