

# Applications of some equivalent quasinorms on sequence spaces

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## Abstract

Let  $x \in \ell_\infty$  be a bounded (real or complex) sequence. We define  $a_n(x) = \inf\{\|x - \bar{x}\| : \text{card}(\bar{x}) < n\}, n = 1, 2, \dots$ , where  $\text{card}(\bar{x}) = \text{card}\{i \in N : \bar{x}_i \neq 0\}$ . Let  $K$  the set of all sequences  $x \in \ell_\infty$  such that  $x_1 \geq x_2 \geq \dots \geq 0$  and  $\text{card}(x) < n < \infty$ . The symmetric norming functions,  $\phi$ , of R. Schatten are defined as follows:  $\phi : K \rightarrow R$  and  $\phi(x) > 0$  if  $x \neq 0$ ;  $\phi(x + y) \leq \phi(x) + \phi(y)$ ;  $\phi(\alpha x) = \alpha\phi(x), \alpha \geq 0$ ;  $\phi(1, 0, 0, \dots) = 1$ ;  $\phi(x) \leq \phi(y)$  if  $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, k = 1, 2, \dots$

We denote by  $\ell_\phi = \{x \in \ell_\infty : \phi(\{a_i(x)\}) < \infty\}$ , where  $\phi(\{a_i(x)\}) = \lim_{n \rightarrow \infty} \phi(a_1(x), a_2(x), \dots, a_n(x), 0, 0, \dots)$ .

We prove that  $\ell_\phi$  is a quasinormed sequence space, where the quasinorm is  $\|x\|_\phi = \phi(\{a_n(x)\})$ .

From the equivalences  $\|x\|_\phi \approx \|x\|_\phi^+ = \phi(\{a_{2n-1}(x)\})$  and  $\|x\|_{\bar{\phi}} \approx \|x\|_{\bar{\phi}}^* = \bar{\phi}(\{x_{n^2}\}), \bar{\phi}(x_i) := \phi(\frac{x_i}{i})$  we obtain some interpolation properties for some quasinormed spaces.

## 1 Introduction

Let  $\ell_\infty$  be the space of all bounded sequences ( $x = (x_n) \in \ell_\infty$  if  $\|x\|_\infty = \sup_n |x_n| < \infty$ ). We denote by  $\text{card}(x) = \text{card}\{n \in N : x_n \neq 0\}$ .

For all  $x \in \ell_\infty$  we define the sequence of the approximation numbers ( $a_n(x)$ ) as follows:

$$a_n(x) = \inf\{\|x - \bar{x}\|_\infty : \text{card}(\bar{x}) < n\}, n = 1, 2, \dots$$

It is obvious that:

$$\|x\|_\infty = a_1(x) \geq a_2(x) \geq \dots \geq 0$$

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The subset  $K \subseteq \ell_\infty$  is defined as follows:

$$K = \{x \in \ell_\infty : \text{card}(x) = n < \infty \text{ and } x_1 \geq x_2 \geq \dots \geq 0\}$$

A function  $\phi : K \rightarrow R$  is called symmetric norming function, [3], [4], [6], if the following conditions are verified:

1.  $\phi$  is a norm on the cone  $K$
2.  $\phi(1, 0, 0, \dots) = 1$
3. If  $x, y \in \ell_\infty$  are such that  $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, k = 1, 2, \dots$ , then  $\phi(x) \leq \phi(y)$ .

Examples of symmetric norming functions are:  $\phi_\infty, \phi_\infty(x) = x_1; \phi_p, \phi_p(x) = (\sum x_i^p)^{\frac{1}{p}}, 1 \leq p < \infty$ .

For the case  $x \in \ell_\infty, x_1 \geq x_2 \geq \dots \geq 0$  and  $\text{card}(x) = \infty$ , we take

$$\phi(x) = \lim_{n \rightarrow \infty} \phi(x_1, x_2, \dots, x_n, 0, 0, \dots).$$

**Definition 1.1** Let  $\phi, \psi$  be symmetric norming functions. The conjugate (dual) of  $\psi$  relative to  $\phi$  is the function:

$$\psi_\phi^*(x) = \sup_{y \in K, y \neq 0} \frac{\phi(xy)}{\psi(y)}, \text{ where } xy := (x_1 y_1, x_2 y_2, \dots).$$

**Definiton 1.2**  $\ell_\phi = \{x \in \ell_\infty : \phi(a_n(x)) < \infty\}$ .

We prove that  $\ell_\phi$  is a quasinormed sequence space.

## 2 Properties of the numbers $a_n(x)$ and sequence space $\ell_\phi$

For to prove that  $\ell_\phi$  is a sequence space it is necessary to investigate some properties of  $a_n(x)$ .

**Proposition 2.1** The numbers  $a_n(x)$  verify the inequality:

$$\sum_{n=1}^k a_n(x_1 + x_2) \leq 2 \sum_{n=1}^k (a_n(x_1) + a_n(x_2)), k = 1, 2, \dots, x_1, x_2 \in \ell_\infty.$$

**Proof.** For  $\epsilon > 0$  there are  $\bar{x}_1, \bar{x}_2 \in \ell_\infty$  such that  $\text{card}(\bar{x}_i) < n, i = 1, 2$ , and  $\|x_i - \bar{x}_i\| \leq a_n(x_i) + \frac{\epsilon}{2}$ .

Since  $\text{card}(\bar{x}_1 + \bar{x}_2) < 2n - 1$ , we obtain:  $a_{2n-1}(x_1 + x_2) \leq \| (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2) \| \leq a_n(x_1) + a_n(x_2) + \epsilon$ .  $\epsilon$  being arbitrary, it results the inequality:  $a_{2n-1}(x_1 + x_2) \leq a_n(x_1) + a_n(x_2)$ .

Now we can write

$$\sum_1^k a_n(x_1 + x_2) \leq \sum_{n=1}^{2k} a_n(x_1 + x_2) = \sum_1^k a_{2n-1}(x_1 + x_2) + \sum_1^k a_{2n}(x_1 + x_2) \leq 2 \sum_1^k (a_n(x_1) + a_n(x_2)), k = 1, 2, \dots$$

This proves the inequality (1).

**Remark** It is known that  $a_n(x_1 x_2) \leq \| x_1 \|_\infty \cdot a_n(x_2)$ , [9], [10] and then it results:

$$\text{For } \lambda \neq 0, a_n(\lambda x) \leq |\lambda| a_n(x) \text{ and } a_n(x) = a_n(\lambda \cdot \frac{1}{\lambda} x) \leq \frac{1}{|\lambda|} a_n(\lambda x).$$

Hence  $a_n(\lambda x) = |\lambda| a_n(x)$ , relation which is obvious true for  $\lambda = 0$ .

Now we prove the following

**Proposition 2.2**  $\ell_\phi$  is a quasinormed sequence space with the quasinorm  $\| x \|_\phi = \phi(a_n(x))$ .

**Proof.**

1. Obvious  $\| x \|_\phi > 0$  if  $x \neq 0$
2. Let  $x_i, i = 1, 2$ , be two sequences from  $\ell_\phi$ . We prove that  $x_1 + x_2 \in \ell_\phi$ .

From proposition 2.1 we obtain:

$$\| x_1 + x_2 \|_\phi = \phi(\{a_n(x_1 + x_2)\}) \leq \phi(\{2(a_n(x_1) + a_n(x_2))\}) \leq 2(\| x_1 \|_\phi + \| x_2 \|_\phi) < \infty.$$

Then  $x_1 + x_2 \in \ell_\phi$  and the property (2) of the quasi-norm  $\| \cdot \|_\phi$  is verified.

3.  $\| \lambda x \|_\phi = |\lambda| \phi(x) < \infty$  if  $\lambda \in R$  and  $x \in \ell_\phi$ .

### 3 Equivalent quasinorms on the spaces $\ell_\phi$

Of great interest will be some equivalent quasinorm on the spaces  $\ell_\phi$ .

We denote:  $\| x \|_\phi^+ = \phi(a_{2n-1}(x))$  and  $\| x \|_\phi^* = \phi(a_{n^2}(x))$ .

**Proposition 3.1** The quasinorms  $\| x \|_\phi^+$  and  $\| x \|_\phi$  are equivalent for all  $\phi$ .

**Proof.**

From the proof of the proposition 2.1 it results that  $\| \cdot \|_\phi \approx \| \cdot \|_\phi^+$ , where  $\| x \|_\phi^+ = \phi(\{a_{2n-1}(x)\})$ , because:

$$\sum_{n=1}^k a_{2n-1}(x) \leq \sum_{n=1}^k a_n(x) \leq \sum_{n=1}^k a_{2n-1}(x) + \sum_{n=1}^k a_{2n}(x) \leq 2 \sum_{n=1}^k a_{2n-1}(x), k = 1, 2, \dots$$

and hence

$$\|x\|_{\phi}^+ \leq \|x\|_{\phi} \leq 2 \|x\|_{\phi}^+, \forall \phi.$$

**Proposition 3.2** The quasinorms  $\|x\|_{\phi}$  and  $\|x\|_{\phi}^*$  are equivalent if the function  $\phi$  is  $\bar{\phi}$ , where  $\bar{\phi} : (x_n) \in K \rightarrow \phi(\alpha_n x_n), 1 = \alpha_1 \geq \alpha_2 \geq \dots \geq 0$  and  $\alpha_{n^2} \leq \frac{c}{n} \alpha_n, \forall n \in N, c$  being constant.

**Proof.**

It is obvious that  $\|x\|_{\bar{\phi}}^* \leq \|x\|_{\bar{\phi}}$ , because  $(a_n(x))$  is decreasing.

From the properties of the sequences  $(\alpha_n)$  and  $(a_n(x))$  we can write:

$$\sum_{n=1}^k \alpha_n a_n(x) \leq \sum_{n=1}^{(k+1)^2-1} \alpha_n a_n(x) = \sum_{n=1}^k \sum_{j=n^2}^{(n+1)^2-1} \alpha_j a_j(x) \leq 3c \sum_{n=1}^k \alpha_n a_{n^2}(x), k = 1, 2, \dots$$

Then it results:

$$\|x\|_{\bar{\phi}} = \phi(\alpha_n a_n(x)) \leq 3c \phi(\alpha_n a_{n^2}(x)) = 3c \|x\|_{\bar{\phi}}^*$$

Hence  $\|x\|_{\bar{\phi}} \approx \|x\|_{\bar{\phi}}^*$ .

**Remark.** In a particular case  $(\alpha_i) = (\frac{1}{i})$ .

## 4 Applications of the equivalence between

$$\|x\|_{\phi}^+ \text{ and } \|x\|_{\phi}^*$$

We denote  $X_{\Sigma} = X_0 + X_1$  and  $Y_{\Sigma} = Y_0 + Y_1$ , where  $(X_0, X_1)$  and  $(Y_0, Y_1)$  are interpolation couples.

If  $(X_0, X_1)$  is an interpolation couple of normed spaces, the interpolation space  $\bar{X} = (X_0, X_1)_{\theta, q}, 0 < q < \infty, \theta \in (0, 1)$ , is defined, [1], [8], [9], [11], as follows:

$$(X_0, X_1)_{\theta, q} = \{x \in X_0 + X_1 : (\int_0^{\infty} [t^{-\theta} K(t, x)]^q \frac{dt}{t})^{\frac{1}{q}} < \infty\},$$

where

$$K(t, x) = \inf_{x_0 + x_1 = x} \{\|x_0\|_{X_0} + t \|x_1\|_{X_1}\}$$

**Definition 4.1** The interpolation couple  $(Y_0, Y_1)$ , where  $Y_0, Y_1$  are normed (Banach) spaces, have the approximation property (H) if exists the constant  $c > 0$  such that for any  $\epsilon > 0$  and any finite sets  $Z_i \subset Y_i, i = 1, 2$ , exist the application  $P \in L(Y_{\Sigma}, Y_{\Sigma})$  for wich  $P|_{Y_0} \in L(Y_0, Z_0), P|_{Y_1} \in L(Y_1, Z_1)$  and more the following properties are true:

1.  $P(Y_i) \subset Y_0 \cap Y_1$ ;
2.  $\|P\|_{L(Y_i)} \leq c$
3.  $\|Px - x\|_{Y_i} \leq \epsilon, \forall x \in Z_i, i = 1, 2$

Let  $T : X \rightarrow Y$  be a linear and bounded operator ( $T \in L(X, Y)$ ), the dyadic entropy numbers are defined as follows:

$$e_n(T) = \inf\{\sigma > 0 : \exists y_1, \dots, y_n \in U_F \text{ such that } TU_X \subseteq \cup_{i=1}^{2^{n-1}} \{y_i + \sigma U_X\},$$

$$\text{where } U_X = \{x \in X : \|x\| \leq 1\}$$

In the paper [11] is proved that, if  $(Y_0, Y_1)$  have the property (H), the following relation is true:

$$e_{m+n-1}(T : (X_0, X_1)_{\theta, q} \rightarrow (Y_0, Y_1)_{\theta, q}) \leq 4 \max\{1, c\} [e_m(T : X_0 \rightarrow Y_0)^{1-\theta} e_n(T : X_1 \rightarrow Y_1)^\theta]$$

Where  $c$  is the constant from the property (H),  $m, n = 1, 2, \dots$

**Corollary 4.1**  $e_{2n-1}(T : (X_0, X_1)_{\theta, q} \rightarrow (Y_0, Y_1)_{\theta, q}) \leq 4 \max\{1, c\} [e_n(T : X_0 \rightarrow Y_0)^{1-\theta} e_n(T : X_1 \rightarrow Y_1)^\theta]$

Let  $L_\phi^{(e)}(X, Y)$  be the entropy ideal ( $L_\phi^{(e)}(X, Y) = \{T : \phi(\{e_n(T)\}) < \infty\}$ )

and let  $\phi_{(p)}(x_n) = (\phi(x_n^p))^{\frac{1}{p}}, 0 \leq p < \infty$ .

From the corollary 4.1 we obtain:

**Proposition 4.2** If  $(X_0, X_1), (Y_0, Y_1)$  are interpolation couples of normed spaces and  $(Y_0, Y_1)$  has the approximation property (H) the following inclusion is true:

$$L_{\psi_{(1-\theta)}}^{(e)}(X_0, Y_0) \cap L_{(\psi_\phi)_{(\theta)}}^{(e)}(X_1, Y_1) \subseteq L_\phi^{(e)}((X_0, X_1)_{\theta, q}, (Y_0, Y_1)_{\theta, q}).$$

for all symmetric norming functions  $\phi$  and  $\psi$ .

**Proof.** From the proposition 1.3 and the corollary 4.1 we can write

$$\begin{aligned} & \phi(\{e_n(T : (X_0, X_1)_{\theta, q} \rightarrow (Y_0, Y_1)_{\theta, q})\}) \approx \\ & \approx \phi(\{e_{2n-1}(T : (X_0, X_1)_{\theta, q} \rightarrow (Y_0, Y_1)_{\theta, q})\}) \leq \\ & \leq 4 \max\{1, c\} \phi(e_n(T : X_0 \rightarrow Y_0)^{1-\theta} \cdot e_n(T : X_1 \rightarrow Y_1)^\theta) \leq \\ & \leq 4 \max\{1, c\} \psi(e_n(T : X_0 \rightarrow Y_0)^{1-\theta}) \cdot \psi_\phi^*(e_n(T : X_1 \rightarrow Y_1)^\theta) < \infty, \end{aligned}$$

which proves the inclusion.

**Remarks** If the particular case of the functions  $\phi_p, 1 \leq p < \infty$ , we obtain the sequence spaces  $\ell_p$  and the entropy ideals

$$L_p^{(e)}(X, Y) = \{T : (\sum e_n^p(T))^\frac{1}{p} < \infty\}.$$

For the ideals  $L_p^{(e)}$  it is known the particular inclusion:

$$L_{p_0}^{(e)}(X_0, Y_0) \cap L_{p_1}^{(e)}(X_1, Y_1) \subseteq L_p^{(e)}((X_0, X_1)_{\theta, q}, (Y_0, Y_1)_{\theta, q}),$$

where  $1 \leq p_0 < p_1 < \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \theta \in (0, 1)$ .

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