On the Postulates for Lattices

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Abstract

This paper is inspired from a seria of papers written by J.A. Kalman, in 1955-1959, having the subject the postulates for lattices. It refers especially to systems formed from absorption, idempotence and associativity laws. Following the Kalman's indications from [1], we studied an extended system of axioms, finding all the implications between the subsets of this system.

Introduction

Let us denote by \mathcal{L} the family of all algebraic systems $l=(L,\wedge,\vee)$ consisting of a set L, together with two binary operations on it and let ω be the following set of axioms:

(1)	$x \land (x \lor y) = x$	(5)	$x \lor (x \land y) = x$
(2)	$x \lor (y \land x) = x$	(6)	$x \land (y \lor x) = x$
(3)	$(y \lor x) \land x$	(7)	$(y \land x) \lor x = x$
(4)	$(x \land y) \lor x = x$	(8)	$(x \lor y) \land x = x$
(A)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(B)	$x \lor (y \lor z) = (x \lor y) \lor z$
(C)	$x \wedge y = y \wedge x$	(D)	$x \lor y = y \lor x$
(I)	$x \wedge x = x$	(J)	$x \lor x = x$

and for each $\xi \in \omega$, let \mathcal{L}_{ξ} be the family of all l in \mathcal{L} such that l obeys all the laws in ξ . Sorkin considered the set ω in [5] §2, and found all the subsets of ω which constitute an independent set of axioms for lattices. For each $\xi \subseteq \omega$, \mathcal{L}_{ξ} , is a family of generalized lattices. Mostly in the years'60 and '70, a few mathematicians studied noncommutative generalizations of lattices: S.I. Matsushuita, P. Jordan, M.D. Gerhardts, H. Alfonz and nowadays J. Leech, R.J.Bignall, Gh. Fărcaş, Matthew Spinks, Karin Cvetko-Vah, From these, Gh. Fărcaş and J. Leech have helped permanently author of this paper in studying these structures.

About the postulates for lattices have also been written books we mention here "Axiomele laticilor şi algebrelor booleene" ("The Axioms of Lattices and Boolean Algebras") by S. Rudeanu.

In studying noncommutative generalizations for lattices, there were considered different systems of axioms included in. The study of the system ω made by Kalman in [1] is useful even today, as much as the study of an enlarged system denoted by ω_+ , having in addition two axioms (A_0) and (B_0) , which are weaker than the associativity of " \wedge " and " \vee ".

Kalman mentioned that, at the beginning of the study of any family \mathcal{L}_{ξ} , the following problems arise: P_{ξ} to find all the subsets of ω that constitute an independent system of axioms for the family \mathcal{L}_{ξ} and Q_{ξ} to find all the laws (X) in ω which are obeyed by every l in \mathcal{L}_{ξ} . Sorkin has solved the problem P_{ω} and the problem Q_{ω} is trivial since \mathcal{L}_{ω} coincide all the class of lattices. Kalman proved results that essesntially solve all the problems P_{ξ} and Q_{ξ} , with $\xi \subseteq \omega$. He gave a table presenting "what results" from each independent subset of ω . The table has 95 lines and contains 351 such implications.

In [2] and [3] we studied problems conected with the problems P_{ξ} and Q_{ξ} with $\xi \subseteq \omega$: the idempotency in systems of the form $\mathcal{L}[_{-} _{-}]$, namely the systems \mathcal{L}_{ξ} where ξ is constituted from two absorption laws, the problem of commutativity of \wedge and \vee in the systems of the form $\mathcal{L}[_{-} _{-} _{-}]$, the relations among the systems of the form $\mathcal{L}[_{-} _{-} _{-}]$. This study was done using direct proofs and counterexamples.

Also Kalman considered weaker axioms than associativity:

$$(A_0)$$
 $x \wedge (y \wedge x) = (x \wedge y) \wedge x$ and (B_0) $x \vee (y \vee x) = (x \vee y) \vee x$

If we add to the system ω these two axioms, then the study of $\omega_+ = \omega \cup \{(A_0), (B_0)\}$ is again interesting. For instance we are interested to find out what absorption, idempotency, commutativity axioms result from a certain system of absorption, idempotency, commutativity axioms. We want to see when the associativity axioms are essential in obtaining a certain result and when can they be replace by weaker axioms. The study of this extended system of axioms was proposed by Kalman in [1] §3 and is made by me in the present paper.

In the first section we make some remarks on the results obtained in [2] and [3] and the results obtained by Kalman concerning the subsets of absorption identities. The second section (Closure operations) presents some elementary results concerning closure operations on a given complete lattice. In this second section we define also the closure operator a, which will be essential for solving the proposed problem.

The third section presents the compatibility of a with a group of automorphisms. The fourth section contains the main results of the paper. The

preparing results which are the four lemmas were given by Kalman in [1] and proved here by me. These helped me to establish Theorem 1 and Theorem 2 for ω_{+} and make the program for proving them.

1 The seven classes of noncommutative lattices

As we said in the previous section, in [2] were presented some classes of algebraic structures generalizing lattices, of the form $\mathcal{L}[_{--}]$, namely \mathcal{L}_{ξ} where ξ is constituted from four absorption laws. More precisely were considered groups of four absorptions, two having the operation \wedge outside the brackets and the other two, the operation \vee outside the brackets. For instance

$$a \wedge (a \vee b) = a$$
, $(a \vee b) \wedge a = a$, $a \vee (a \wedge b) = a$, $(b \wedge a) \vee a = a$.

We attempted to answer how many groups of such absorption laws we have, namely how many essentially different classes of noncommutative generalizations of lattices define they. We considered the following relation between two systems of the described type:

$$S_1 \sim S_2 \Leftrightarrow S_1 = S_2 \text{ or } (S_1)^{\land} = S_2 \text{ or } (S_1)^{\lor} = S_2 \text{ or } (S_1)^{\land\lor} = S_2.$$

that is: S_1 is equivalent with S_2 iff S_1 is equal to S_2 or S_2 can be obtained from S_1 by interchanging the performing order of " \wedge ", or " \vee ", or of both operations. I found twelve equivalence classes having the representatives: $S'_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\vee}, S_{\vee}, S_{\vee}, S_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\vee}, S_{\vee}, S_{\wedge}$ (see tabel 1).

 $S'_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\vee}, S_{\vee}, S_{\vee}, S_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\wedge}, S_{\vee}, S_{\vee}, S_{\vee}, S_{\wedge}, S_{\vee}, S_{\vee}, S_{\wedge}, S_{\vee}, S_{\vee},$

Kalman in his paper [1] considered different the equivalence relation between the systems of axioms:

$$S_1 \sim S_2 \Leftrightarrow \exists \ \sigma \in P \text{ such that } S_2 \text{ is the image of } S_1 \text{ by } \sigma.$$

Here P is a group of permutations generated by two certain permutation of the axioms from ω . In fact, the Kalman's definition can be described thus:

$$S_1 \sim S_2 \Leftrightarrow S_1 = S_2 \text{ or } (S_1)^{\wedge} = S_2 \text{ or } (S_2)^{\wedge} \text{ or } (S_1)^{\wedge\vee} = S_2 \text{ or } (S_1^*)^{\wedge} = S_2 \text{ or } (S_2^*)^{\wedge} \text{ or } (S_1^*)^{\wedge\vee} = S_2$$



For instance, if we want to see the equivalence class of the system formed from the axioms (1) and (2), system denoted by [12], we know that the axiom (1) can become any other absorption axiom, and also (2) by the following sketch:

Thus, the equivalence class of [12] is [12] = {[12], [85], [34], [67], [56], [41], [78], [23]}. If we examine the Kalman's table 2 from [1], he obtained 12 equivalence classes having representatives with four absorption laws. We present below an extract from this table, containing them:

1.
$$1234$$
 $\overline{S'_{\wedge\vee}}$ and $\overline{S_{\wedge\vee}}$ 3. 1236 of type $3+1$ 4. 1237 $\overline{S_{\wedge\vee}}$ and $\overline{S_{\wedge\wedge}}$ 5. 1256 $\overline{S'_{\wedge\vee}}$ and $\overline{S_{\wedge\wedge}}$ 6. 1257 of type $3+1$ 7. 1258 $\overline{S_{\wedge}}$ and $\overline{S'_{\wedge\wedge}}$ 8. 1267 $\overline{S'_{\vee}}$ and $\overline{S'_{\wedge\wedge}}$ 9. 1268 of type $3+1$ 10. 1278 $\overline{S_{\wedge\vee}}$ 11. 1357 $\overline{S_{\wedge\vee}}$ 12. 1368 of type $3+1$

Table 1

From these twelve, four are of type 3+1 (having three absorption with the same operation outside the brackets). The other eight are representatives for the equivalence classes we found, in [2], as it is indicated in table 1. As it is mentioned in all the papers [1], [2], [3] there are examples that prove that these classes of noncommutative generalizations are distinct.

2 Closure operations

Let us consider an operator $a: \mathcal{P}(\omega_+) \to P(\omega_+)$ defined by:

$$a\xi = \{(X) \in \omega_+ \mid (X) \text{ is obeyed in any } l \in \mathcal{L}_{\xi}\}.$$

We can easily verify that it is a closure operator on $\omega_+ = \omega \cup \{A_0, B_0\}$ (defined in the Introduction). In order to establish the exact value of $a\xi$, for every $\xi \subseteq \omega_+$, we will need other closure operators, which "approximate" up and down our operator a.

Our discussion from this section takes place in the general frame of a complete lattice with greatest element V, and it will be applied in the next sections for the complete lattice of systems of axioms $\mathcal{P}(\omega_+)$.

Following the Kalman ideas, we will consider a complete lattice L with greatest element V, G a group of lattice automorphisms $g: L \to L$ and \mathcal{C} the set of all closure operations $c: L \to L$, which are "compatible with G", i.e. which are such that xgc = xcg, for all x and g in G. If we define an order relation " \leq " by: $c \leq c'$ if and only if $xc \leq xc'$ for all x in L, it is easy to verify that \mathcal{C} becomes a complete lattice. Let \mathcal{Z} be the set of all subsets Z of L which are such that (i) $V \in Z$ (ii) if $x \subseteq Z$, then $Inf X \in Z$ and (iii) if $x \in Z$, then $xg \in Z$ for all g in G. The set Z becomes a complete lattice when for Z, Z' in Z we set $Z \subseteq Z'$ if and only if $Z \subseteq Z'$ (set theoretic inclusion). Also a dual isomorphism ψ of G onto Z may be defined by setting: $c\psi = \{x \mid x \in L \text{ and } x = xc\}$.

The inverse dual isomorphism ψ is given by:

$$x(Z\psi^{-1}) = Inf\{y \mid y \in Z \quad \text{and} \quad y \ge x\}$$

If c_0 is a partially defined unary operation on L i.e. a mapping of some subset L_0 of L into L, and if c in C is given by:

$$c = \text{Inf}\{b \mid b \in \mathcal{C} \text{ and } xbc_0 \text{ for all } x \in L_0\},\$$

it can be easy verified that $c: L \to L$ is a closure operator. We will call c_0 a "G-support" of c. If Z_0 is any subset of L, and if Z in Z is given by:

$$Z = Inf\{W \mid W \in \mathcal{Z} \text{ and } W \supseteq Z_0\},$$

we will call Z_0 a "G base" of the closure operation $Z\psi^{-1}$. If c is any closure operation on L and $x \in L$, we will say that x is "c"-closed if x = xc and that x is "c-independent" if no y in L is such that y < x and yc = xc.

Remark 1. If $c_1, c_2 : L \to L$ are two closure operation such that $c_1 \leq c_2$, then, for any $\xi \in L$,

- a) ξ is c_1 -dependent $\Rightarrow \xi$ is c_2 -dependent
- b) ξ is c_2 -dependent $\Rightarrow \xi$ is c_1 -independent.

Indeed, suppose there exist $y < \xi$ such that $yc_1 = \xi c_1$. Then we have $y < \xi \le \xi c_1 = yc_1 \le yc_2$, thus $\xi \le yc_2$. Applying c_2 we have $\xi c_2 \le yc_2$. The converse, inequality is obvious since $y < \xi$. Thus $\xi c_2 = yc_2$ and ξ is c_2 dependent

b) Results from a).

3 The compatibility of a with a group of lattice - automorphisms

On the family \mathcal{L} of all algebraic systems $l=(L,\wedge,\vee)$ consisting of all the set L together with binary operations \wedge and \vee , Kalman considered the transformations Π and ρ :

$$\pi: \mathcal{L} \to \mathcal{L}, \ \rho: \mathcal{L} \to \mathcal{L}, \ \forall \ l = (L, \wedge, \vee) \in \mathcal{L}, \ l\Pi = (L, \wedge_{\Pi}, \vee_{\Pi}), \ l\rho = (L, \wedge_{\varrho}, \vee_{\varrho})$$

where,

- (9) $x \wedge_{\pi} y = y \vee x$, $x \vee_{\pi} y = x \wedge y$, $\forall x, y \in L$
- (10) $x \wedge_{\rho} y = x \vee y$, $x \vee_{\rho} y = x \wedge y$, $\forall x, y \in L$.

It is easy to verify that $\rho\pi = \pi^3\rho$ and $\Pi^4 = \rho^2 = \varepsilon$ (the identity transformation). The transformation Π and ρ generate in the subgroup of all transformations on L, a subgroup Γ and all the elements of Γ can be written in at least one way in the form $\Pi^m\rho^n$, $m \in \{0, 1, 2, 3\}$, $n \in \{0, 1, 2\}$.

Kalman also considered two permutations p and q in the permutation group of the elements $1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, I, J, A_0, B_0$. We will consider in the same way two permutation p and q of the elements $1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, I, J, A_0, B_0$:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 2 & 3 & 4 & 1 & 8 & 5 & 6 & 7 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$

By the fact p correspond to the permutation Π we meant that p indicates the correspondence between the axioms fulfilled in an algebraic structure $l \in \mathcal{L}$ and the correspondent axioms that hold in $l\Pi$.

Analogously we determine q. It is easy to verify that, for all $l \in \mathcal{L}$ and $(X) \in \omega_+$; l obeys $(X) \Leftrightarrow l\Pi$ obeys $(X)p \Leftrightarrow l\rho$ obeys (X)q. Let P be the subgroups generated by p and q in the group of all permutations of the elements of ω_+ . It is easily seen that the elements of p and q verify: $p^4 = q^2 = e$ and $qp = p^3q$.

If follows that the elements of p are precisely of the form p^mq^n , $m \in \{0,1,2,3\}$, $n \in \{0,1,2\}$ and that the mapping $\lambda : P \to \Gamma$, $\lambda(p^m\rho^n) = \Pi^m\rho^n$, $m \in \{0,1,2,3\}$, $n \in \{0,1,2\}$ is an homomorphism of P onto Γ (we will see in §4 that λ is in fact an isomorphism).

We give below the permutation subgroup generated by p and q

r	1	2	3	4	5	6	7	8	A	B	C	D	I	J	A_0	B_0
\overline{e}	1	2	3	4	5	6	7	8	A	B	C	D	I	J	A_0	B_0
\overline{p}	2	3	4	1	8	5	6	7	В	A	D	C	J	I	B_0	A_0
p^2	3	4	1	2	7	8	5	6	A	B	C	D	I	J	A_0	B_0
p^3	4	1	2	3	6	7	8	5	В	A	D	C	J	I	B_0	$\overline{A_0}$
\overline{q}	5	6	7	8	1	2	3	4	B	A	D	C	J	I	B_0	A_0
\overline{pq}	6	7	8	5	4	1	2	3	A	B	C	D	I	J	A_0	B_0
p^2q	7	8	5	6	3	4	1	2	В	A	D	C	J	I	B_0	A_0
p^3q	8	5	6	7	2	3	4	1	A	B	C	D	I	J	A_0	B_0

Table 2.

Using the following:

- an axiom (X) is true in $l \Leftrightarrow$ the axiom (X)p is true in $l\Pi = l(p\lambda)$
- an axiom (X) is true in $l \Leftrightarrow$ the axiom (X)q is true in $l\rho = l(q\lambda)$
- λ is a homomorphism,

the following lemma hold:

Lemma 1. For all $l \in \mathcal{L}$, (X) in ω_+ and $r \in P$, l obeys the law (X) if and only if $l(r\lambda)$ obeys the law (X)r.

If we choose a permutation $r \in P$, to each subset of axioms from ω_+ we can associate the corresponding subset of axioms, by r. Thus we have defined a transformation μ :

$$\xi(r\mu) = \{(Y) \mid \exists \ r \in P \text{ and } \exists (X) \in \xi \text{ and such that } (Y) = (X)r\}$$

having the domain P. $r\mu$ is a lattice automorphism of the Boolean algebra $P(\omega_+)$. If we denote by G the immage of μ , we have that μ is an isomorphism of P onto the group of automorphisms of $P(\omega_+)$.

Let's choose again a permutation $r \in P$. We notice that the operator a defined in §2 is, by it's definition, compatible with any transformation which associates to a $\xi \subseteq \omega_+$ the resulting subset $(\xi)r$ (immage of the set ξ by

r), namely $[(\xi)r]a = (\xi a)r$. On the other side, from definition of μ we have $\xi(r\mu) = (\xi)r$. Thus,

$$[\xi(r\mu)]a = [(\xi)r]a = (\xi a)r = (\xi a)(r\mu), \ \forall \ r\mu \in G,$$

namely we have:

Lemma 2. The closure operation a is compatible with G.

In the final of this paragraph we will define on ω_+ the relation: if $\xi, \eta \subseteq \omega_+$ are such that $\eta = \xi(r\mu)$ for some $r \in P$. we will call ξ and η "congruent" subsets of ω_+ and it follows from Lemma 2 that, if a subset ξ of ω_+ is a closed, [a-independent] then every η congruent to ξ is a-closed [a-independent].

4 Main result

If a partially defined unary operation c_0 on a given set has domain $\{\xi_1, \xi_2, \dots, \xi_n\}$ and we have $\xi_i c_0 = \eta_i$, $i = \overline{1, n}$, we will say that c_0 has "defining relations" $\xi_1 \to \eta_1, \xi_2 \to \eta_2, \dots, \xi_n \to \eta_n$. If the distinct elements of a nonempty subset ξ of ω are $(X_1), \dots, (X_n)$, we will write $\xi = [X_1 X_2 \dots X_n]$. Let a_0 be the partially defined operation on the subset of ω which has defined relations

$$[A] \to [A_0], [C] \to [A_0], [12] \to [J], [15] \to [J], [1C] \to [8], [1D] \to [6],$$

 $[17] \to [I], [123] \to [8], [127A_0] \to [8], [1267B_0] \to [D]$ and $[1368BJ] \to [D],$

and let a_1 be the closure operation on the subsets of ω which has G-support a_0 . The following lemma hold:

Lemma 3. $a_1 \leq a$.

Proof. It is sufficient to prove $\xi a \supseteq \xi a_0$ for each ξ in the domain of a_0 .

In the presence of associativity (A) or commutativity (C), the axiom A_0 : $(x \wedge y) \wedge x = x \wedge (y \wedge x)$ is obviously fulfilled. The following six implications and the last one are true by Lemma 3 from [1]. We must prove $[127A_0] \rightarrow [8]$ and $[1267B_0] \rightarrow [D]$.

First we must prove using (1), (2), (7) and (A_0) that (8): $(x \lor y) \land x = x$ is fulfilled. From [12] results [J] and from [1J] results (I). In any $l = (L, \land, \lor)$, from $\mathcal{L}[127A_0]$, for any $x, y \in L$, using A_0 we have first:

$$[x \land (x \lor y)] \land x = x \land [(x \lor y) \land x]$$

The member from left is equal to x by 1. After that we apply $\vee[(x\vee y)\wedge x]$ and thus:

$$x \vee [(x \vee y) \wedge x] = (x \vee y) \wedge x.$$

But by (2) the member from left is equal to x, and thus we obtain that (8) is true in L.

We will prove $[1267B_0]$ implies [D].

$$x \vee y \stackrel{7}{=} [y \wedge (x \vee y)] \vee (x \vee y) \stackrel{6}{=} y \vee (x \vee y) \stackrel{B_0}{=} (y \vee x) \vee y.$$

Analogously $y \lor x = (x \lor y) \lor x$.

Using these, we have:

$$x \vee y = (y \vee x) \vee y = (y \vee x) \vee [y \wedge (y \vee x)] \stackrel{2}{=} y \vee x.$$

Remark 2. The defining relations $[1267B] \rightarrow [D]$ and $[127A] \rightarrow [8]$ from the study of ω , in [1], were replaced, after Kalman's idea with $[1267B_0] \rightarrow [D]$ and $[127A_0] \rightarrow [B]$. Thus the associativity appears just in two of the defining relations of a_0 .

Remark 3. We remark that the value of the operator a_1 can be calculated. We consider the defining relations of a_0 and their permutation obtained by table 2. There are 66 distinct relations, the set of which will be denoted by S. We consider then the operator $c: P(\omega_+) \to P(\omega_+)$ which acts as follows on a given $\varepsilon \subseteq \omega_+$: adds the conclusion of each from the 66 relations, if the respective hypothesis is found in ξ , replacing after that each time ξ with the result system. It's obvious that, there exits a natural number which depends of the given $\xi, n(\xi)$ such that $c^{n(\xi)}(\xi) = c^{n(\xi)+1}(\xi)$. If we consider $n = \sup_{\xi \in \mathcal{L}} n(\xi)$, then , for any $\xi \subseteq \omega_+$

$$\xi \in P(\omega_+)
(11) \xi a_1 = \xi c^n$$

since c^n is a closure operator, verifies $c^n \ge a_0$ on the domain of a_0 and it is the least with these properties.

We will consider now a few algebraic structures (L, \wedge, \vee) which will be counterexamples for certain implications between the subsets of ω_+ . Kalman indicated in [1] the following examples (the sequences mean the rows of the

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corresponding multiplication tables for \wedge and \vee):

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L_1 = \{0, 1, 2, 3, 4\}, 00000 01111 01202 01033 01234 01234 11234 22244 33434 44444 \\ L_2 = \{0, 1, 2, 3, 4\}, 00000 01111 01202 01133 01234 01234 11234 22244 33434 44444 \\ L_3 = \{0, 1, 2\}, 000 011 012 022 112 222 \\ L_4 = \{0, 1\}, 01 01 00 11 \\ L_5 = \{0, 1, 2\}, 000 012 012 022 111 222 \\ L_6 = \{0, 1, 2\}, 010 011 012 022 212 222 \\ L_7 = \{0, 1\}, 01 01 00 11 \\ L_8 = \{0, 1\}, 01 10 00 00 \\ L_9 = \{0, 1\}, 01 11 00 00
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 $L_{10} = \{0, 1, 2\},\ 000\ 011\ 012\ 022\ 212\ 222$

 $L_{11} = \{0, 1, 2\}, 000 011 012 012 222 222$

 $L_{12} = \{0, 1, 2\},\ 000\ 001\ 002\ 012\ 222\ 222$

 $L_{13} = \{0, 1, 2, 3\},\ 0000\ 0111\ 0122\ 0123\ 0233\ 3133\ 3323\ 3333$

Remark 4. a) L_2 doesn't verify A_0 . Indeed for x = 2 and y = 3, $(x \wedge y) \wedge x = x \wedge (y \wedge x) \Leftrightarrow 0 = 1$.

- b) L_3 doesn't verify B_0 . Indeed, for x = 1, y = 0, we have $(x \vee y) \vee x = x \vee (y \vee x) \Leftrightarrow 1 = 2$.
- c) L_{14} satisfies $[1368ACIJA_0B_0]$ and doesn't satisfies the rest of the axioms ω_+ . Indeed (L_{14}, \wedge) is the restrictive semigroup of the chain $0 \le 1 \le 2 \le 3$ and it verifies $[ACIA_0]$. The rest it is easy to verify.

Let z_0 be the following family of subsets of ω_+ .

 $\aleph_{1} = [12345678BCDIJA_{0}B_{0}]$ $\aleph_{2} = [12345678BDIJB_{0}]$ $\aleph_{3} = [123568ACIJA_{0}]$ $\aleph_{4} = [123578ABIJA_{0}B_{0}]$ $\aleph_{5} = [12358ABIJA_{0}B_{0}]$ $\aleph_{6} = [123678ABDIJA_{0}B_{0}]$ $\aleph_{7} = [1258ABCIJA_{0}B_{0}]$ $\aleph_8 = [1368ABCDA_0B_0]$ $\aleph_9 = [1368ABCDIA_0B_0]$ $\aleph_{10} = [1368ABCDIJA_0B_0]$ $\aleph_{11} = [1368ABCIA_0B_0]$ $\aleph_{12} = [17ABA_0B_0]$ $\aleph_{13} = [1368ACIJA_0B_0]$

and let a_2 be the closure operation on the subsets of which has G-base Z_0 .

Lemma 4. $a \leq a_2$.

Proof. Since Z_0 is a G base of a_2 , we have:

$$a_2\psi = Z = \text{Inf}\{W \mid W \in 3\mathcal{Z} \text{ and } W \supseteq Z_0\}$$

and in the same time we know from the definition of ψ that Z is the set of all elements from $P(\omega_+)$ which are a_2 -closed, thus, for any $\xi \subseteq \omega_+$,

$$(12) \xi a_2 = \cap \{ y \in Z \mid y \supseteq \xi \}$$

We have to prove $\xi a \leq \cap \{y \in Z \mid y \supseteq \xi\}, \ \forall \ \xi \subseteq \omega_+$.

We notice from (12) and from definition of \mathcal{Z} that

$$Z = Z_0 \cup \bigcup_{g \in G} g(Z_0) \cap \{\omega_+\} \cup F,$$

where F denotes the sets of the form inf X, where $X \subseteq Z_0 \cup \bigcup_{g \in G} g(Z_0) \cup \{\omega_+\}$.

Thus

(13)
$$\xi a_2 = \bigcap \{ y \in Z_0 \cup \bigcup_{g \in G} g(Z_0) \cup \{\omega_+\} \mid y \supseteq \xi \}$$

If we prove that all the elements considered in the last expression are a-closed, then the lemma is true, since each such y will satisfy

$$y = ya \supseteq \xi a.$$

The element ω_+ is obviously a-closed, and if the elements of Z_0 are a-closed, then also the elements of $g(Z_0)$ are a-closed, since a is compatible with any $g \in G$. The fact that the elements of Z_0 are a-closed results from the fact that the algebraic structures L_i , $i = \overline{1, 13}$ presented in this paragraph verify $[\aleph_i]$ but don't verify $\omega_+ \setminus \aleph_i$.

Remark 5. We remark that the values of the operator a_2 can be calculated. If we consider the systems $\aleph_1, \aleph_2, \ldots, \aleph_{13}$ and the permutated systems obtained from them by table 2, we obtain 104 systems $\aleph_1, \aleph_2, \ldots, \aleph_{104}$. By definition of G (see §3) we have then:

$$(14) \ \xi a_2 \cap \{ y \in \{\aleph_1, \aleph_2, \dots, \aleph_{104}\} \mid y \supseteq \xi \}$$

We now state the theorems which are the main results of this paper.

Theorem 1. The operation a_0 is a G-support of a and the family Z_0 is a G base of a.

Proof. To prove theorem 1 it will be sufficient to show that $a_1 = a_2$. Then, by Lemma 3 and 4, we will have $a_1 = a = a_2$, completing the proof.

Let us consider $\theta \subseteq \omega_+$, arbitrary systems of axioms.

The computer programm calculates θa_1 and θa_2 by (11) and (13) and compares the results. For any $\theta \subseteq \omega_+$ we have $\theta a_1 = \theta a_2$ and thus $a_1 = a = a_2$.

Let's denote by $\overline{\theta}$ the common value of θa_1 and θa_2 .

Theorem 2. A subset of ω_+ is a-independent if and only if it is congruent to one of the subsets θ listed in column θ of table 3. The entry in the row of a certain θ and column $\overline{\theta}$ of table 3 is θa .

Note It may easily be checked that each entry in column θ of table 3 is the lexicographically first element of it's congruence class. Thus, no two subsets in column 0 are congruent to each other.

Proof. Since for each $\theta \subseteq \omega_+$ we know the exact value of θa as we explained in the proof of theorem 1 means that we can establish, by a procedure with an element $\theta \subseteq \omega_+$ is a-independent. Calculating the value of a for the subsystems of θ .

5 About the programm

The programm has got a few functions. The must important are:

- a function "minim" which receive a sequence representing a system of axioms, calculates the elements from the same congruence class using table 2, and returns the least sequence in lexicographical order.
- a function "aplică t" which calculates a_1 for a given sequence θ , using (11).
- a function "aplicaă 2" which calculates a_2 for a given sequence θ , using (13). The matrix \aleph having the rows $\aleph_1, \ldots, \aleph_{104}$ is taken from the main programm and it is generated using table 2 by another function
- a function "verifindep" which verifies if a given system is a-independent.

The main programm generates the subsets of ω_+ in lexicographical order. When each system θ is formed, it is "minimized", by the function "minim". After that the programm verifies the independency of the resulted system, called "min." We can renounce first at verifying the independency and we want to see first that $a_1 = a_2$ as we explained in the proof of the Theorem 1. In this way we follow the logic order of the ideas. In the case when we verify the independency,

when the system θ is independent it is put in a list, introducing it where it is it's place in lexicographical order. After the last element $\xi \subset \omega_+$ has been generated (this is $[B_0]$) and it is verified it's independency, the programm starts to print the list of a-independent systems. After printing $\theta \subseteq \omega_+$ from the list, it calculates θa_1 by the function "aplicat" and θ_2 by the function "aplica 2", verifies if θa_1 and θa_2 coincide and if not, it gives us a message and stops running. No such message has been received and thus $\theta a_1 = \theta a_2$.

In the case of the coincidence, it prints θa_1 and goes further to the next θ from the list.

The table 3, of the results, contains 599 rows for all the 599 independent systems found in ω_{+} .

For a simple writing of the results, the programm prints the letters "k" and "l" instead of notations " (A_0) " and " (B_0) ". The axioms (A), (B), (C), (D), (I), (J) are denoted in the list of the results with small letters.

Let's interpret the results for two systems of axioms.

The system [1234]:

- -is independent since appears in the first coloumn of the tabel 3.
- implies the axioms 1, 2, 3, 4, 5, 6, 7, 8, I, J and no other axioms from ω_+ .
- together with the axiom (A), implies $[12345678ACIJA_0]$.
- -together with the axiom (C), implies $[12345678CIJA_0]$.
- -together with the axiom (A_0) (written in the tabel as k), implies $[12345678CIJA_0]$ namely the same system of axioms as in the case if we had added (C), and the same system of absorption, commutativity, and idempotence axioms as in the case if we had added (A).

The system [1368]:

- -is independent since appears in the first coloumn of the tabel 3.
- -implies the axioms 1, 3, 6, 8 and no other axioms from ω_{+} .
- together with the axiom (A) it implies $[1368AA_0]$.
- together with the axiom (C) it doesn't form an independent subsystem and we must find a subsystem of [1368C] which is independent and implies these axioms. We find [13C], which implies $[1368CA_0]$ and no other axioms beside these.
- -together with the axiom (A_0) , it forms an independent system, and it implies just the same system [1368 A_0]. Thus, from the point of view of

absorption, commutativity and idempotence axioms that result, we have the same result if we add (A) or (A_0) , but we don't have the same result if we add (C) or (A_0) .

tetha	tetha-bar
1	1
12	12ij
123	1238ij
1234	12345678ij
1234a	12345678acijk
1234ab	12345678abcdijkl
1234al	12345678acdijkl
1234c	12345678cijk
1234k	12345678cijk
1234kl	12345678cdijkl
1235	12358ij
12356	123568ij
123567	12345678ij
12356a	123568acijk
12356ab	12345678abcdijkl
12356al	12345678acdijkl
12356b	12345678bdijl
12356bk	12345678bcdijkl
12356k	123568cijk
12356kl	12345678cdijkl
123561	12345678dijl
12357	123578ij
12357a	123578aijk
12357ab	123578abijkl
12357al	123578aijkl
12357b	123578bijl
12357bk	123578bijkl
12357k	123578ijk
12357kl	123578ijkl
123571	123578ijl
1235a	12358aijk
1235ab	12358abijkl
1235al	12358aijkl
1235b	12358bijl
1235bd	12345678bdijl
1235bk	12358bijkl
1235d	12345678dijl
1235k	12358ijk
1235kl	12358ijkl
12351	12358ijl
1236	12368ij
12367	123678ij
1236a	12368aijk
1236ab	123678abdijkl

1236al 123678adijkl 1236b 123678bdijl 1236bk 123678bdijkl 1236k 12368ijk 1236kl 123678dijkl 12361 123678dijl 1237 12378ij 1237a 12378aijk 1237ab 12378abijkl 1237ac 12345678acijk 1237al 12378aijkl 1237b 12378bijl 1237bk 12378bijkl 1237c 12345678cijk 1237k 12378ijk 1237kl 12378ijkl 12371 12378ijl 123a 1238aijk 123ab 1238abijkl 123abc 12345678abcdijkl 123ac 123568acijk 123acl 12345678acdijkl 123al 1238aijkl 123b 1238bijl 123bc 12345678bcdijkl 123678bdijl 123bd 123bk 1238bijkl 123c 123568cijk 123cl 12345678cdijkl 123d 123678dijl 1238ijk 123k tetha tetha-bar 123kl 1238ijkl 1231 1238ijl 125 125ij 1256 1256ij 1256a 1256aijk 1256ab 1256abijkl 1256aijkl 1256al 1256k 1256ijk 1256kl 1256ijkl 1257 1257ij 1257a 12578aijk 1257ab 12578abijkl 1257al 12578aijkl 1257b 1257bijl 1257bk 12578bijkl 1257k 12578ijk 1257kl 12578ijkl

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12571
             1257ijl
1258
             1258ij
1258a
             1258aijk
1258ab
             1258abijkl
1258al
             1258aijkl
1258b
             1258bijl
1258bd
             12345678bdijl
1258bk
             1258bijkl
1258d
             12345678dijl
1258k
             1258ijk
1258kl
             1258ijkl
12581
             1258ijl
125a
             125aijk
125ab
            125abijkl
125abd
            12345678abcdijkl
125ad
            12345678acdijkl
125al
            125aijkl
125b
             125bijl
125bd
             124567bdijl
125bdk
             12345678bcdijkl
125bk
             125bijkl
125d
             124567dijl
125dk
             12345678cdijkl
125k
             125ijk
125kl
             125ijkl
1251
             125ijl
126
             126ij
1267
             1267ij
1267a
             123678aijk
1267ab
             123678abdijkl
1267ac
             12345678acijk
1267al
             123678adijkl
1267b
             1267bdijl
             123678bdijkl
1267bk
1267c
             12345678cijk
1267k
             123678ijk
1267kl
             123678dijkl
12671
             1267dijl
1268
             1268ij
1268a
             1268aijk
1268ab
             1268abijkl
1268al
             1268aijkl
1268b
             1268bijl
1268bk
             1268bijkl
1268k
             1268ijk
1268kl
             1268ijkl
12681
             1268ijl
126a
             126aijk
126ab
             126abijkl
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126abc 12345678abcdijkl 126ac 123568acijk 126acl 12345678acdijkl 126al 126aijkl 126b 126bijl 12345678bcdijkl 126bc 126bk 126bijkl 126c 123568cijk 126cl 12345678cdijkl teha tetha-bar 126k 126ijk 126kl 126ijkl 1261 126ijl 127 127ij 1278 1278ij 127a 1278aijk 127ab 1278abijkl 127ac 124578acijk 127al 1278aijkl 127b 127bijl 127bk 1278bijkl 127c 124578cijk 127k 1278ijk 127kl 1278ijkl 1271 127ijl 128 128ij 128a 128aijk 128ab 128abijkl 128al 128aijkl 128b 128bijl 128bd 123678bdijl 128bk 128bijkl 128d 123678dijl 128k 128ijk 128kl 128ijkl 1281 128ijl 12a 12aijk 12ab 12abijkl 12abc 1258abcijkl 12abcd 12345678abcdijkl 12abd 123678abdijkl 12ac 1258acijk 12345678acdijkl 12acd 12acl 1258acijkl 12ad 123678adijkl 12al 12aijkl 12b 12bijl 12bc 1258bcijkl 12bcd 12345678bcdijkl

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12bd
             1267bdijl
12bdk
             123678bdijkl
12bk
             12bijkl
12c
             1258cijk
             12345678cdijkl
12cd
12cl
              1258cijkl
12d
              1267dijl
12dk
              123678dijkl
12k
              12ijk
12kl
              12ijkl
121
             12ijl
13
             13
135
              135ij
1357
             1357ij
1357a
             1357aijk
             1357abijkl
1357ab
1357ac
             12345678acijk
1357al
              1357aijkl
1357c
              12345678cijk
1357k
              1357ijk
1357kl
             1357ijkl
135a
              135aijk
135ab
              135abijkl
135abc
              12345678abcdijkl
135ac
              123568acijk
135acl
              12345678acdijkl
135al
              135aijkl
135b
              135bijl
135bc
              12345678bcdijkl
135bd
              134568bdijl
135bk
              135bijkl
135c
              123568cijk
135cl
             12345678cdijkl
135d
             134568dijl
135k
             135ijk
135kl
             135ijkl
tetha
            tetha-bar
1351
             135ijl
136
             136
1368
             1368
1368a
              1368ak
1368ab
              1368abkl
1368abi
             1368abikl
1368abj
              1368abdijkl
              1368aik
1368ai
              1368aikl
1368ail
1368aj
              1368aijk
1368ajl
              1368aijkl
1368al
              1368akl
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1368b 1368bl 1368bi 1368bil 1368bik 1368bikl 1368bj 1368bdijl 1368bjk 1368bdijkl 1368bk 1368bkl 1368i 1368i 1368ik 1368ik 1368ikl 1368ikl 1368il 1368il 1368ij 1368j 1368jk 1368ijk 1368jkl 1368ijkl 1368jl 1368ijl 1368k 1368k 1368kl 1368kl 13681 13681 136a 136ak 136ab 136abkl 136abi 136abikl 136abj 136abijkl 136ai 136aik 136ail 136aikl 136aj 136aijk 136ajl 136aijkl 136al 136akl 136b 136bl 136bi 136bil 136bik 136bikl 136bj 136bijl 136bjk 136bijkl 136bk 136bkl 136i 136i 136ik 136ik 136ikl 136ikl 136il 136il 136j 136ij 136jk 136ijk 136jkl 136ijkl 136jl 136ijl 136k 136k 136kl 136kl 1361 1361 13a 13ak 13abkl 13ab 1368abckl 13abc 13abci 1368abcikl 1368abcdijkl 13abcj 13abd 1368abdkl

13abdi 1368abdikl 13abdj 1368abdijkl 13abi 13abikl 13abj 13abijkl 13ac 1368ack 1368acik 13aci 13acil 1368acikl 13acj 1368acijk 13acjl 1368acijkl 13acl 1368ackl 13ad 1368adkl 13adi 1368adikl 13adj 1368adijkl 13ai 13aik tetha tetha-bar 13ail 13aikl 13aj 13aijk 13aijkl 13ajl 13al 13akl 13b 13bl 13bc 1368bckl 13bci 1368bcikl 13bcj 1368bcdijkl 1368bdl 13bd 13bdi 1368bdil 13bdik 1368bdikl 1368bdijl 13bdj 1368bdijkl 13bdjk 13bdk 1368bdkl 13bi 13bil 13bik 13bikl 13bj 13bijl 13bjk 13bijkl 13bk 13bkl 13c 1368ck 13ci 1368cik 13cil 1368cikl 1368cijk 13cj 1368cijkl 13cjl 1368ckl 13cl 13d 1368dl 13di 1368dil 13dik 1368dikl 13dj 1368dijl 13djk 1368dijkl 13dk 1368dkl 13i 13i 13ik 13ik 13ikl 13ikl

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13il
              13il
              13ij
13 j
              13ijk
13jk
13jkl
              13ijkl
13jl
              13ijl
13k
              13k
13kl
              13kl
              131
131
15
              15ij
15a
              15aijk
15ab
              15abijkl
15abc
              1258abcijkl
15abcd
              12345678abcdijkl
15ac
              1258acijk
15acd
              12345678acdijkl
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              1258acijkl
15ad
              1456adijkl
15al
              15aijkl
15c
              1258cijk
15cd
              12345678cdijkl
15cl
              1258cijkl
15k
              15ijk
15kl
              15ijkl
16
              16
              16ak
16a
16ab
              16abkl
              1368abckl
16abc
16abci
              1368abcikl
16abcj
              1368abcdijkl
16abi
              16abikl
16abj
              16abijkl
16ac
              1368ack
16aci
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16acil
              1368acikl
              1368acijk
16acj
16acjl
              1368acijkl
16acl
              1368ackl
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              16aik
16ail
              16aikl
              16aijk
16aj
16ajl
              16aijkl
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             tetha-bar
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16b
              16bl
              1368bckl
16bc
              1368bcikl
16bci
16bcj
              1368bcdijkl
              16bil
16bi
16bik
              16bikl
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16bj
             16bijl
16bjk
             16bijkl
16bk
             16bkl
16c
             1368ck
16ci
             1368cik
             1368cikl
16cil
16cj
             1368cijk
16cjl
             1368cijkl
             1368ckl
16cl
16i
             16i
16ik
             16ik
16ikl
             16ikl
             16il
16il
             16ij
16 j
16jk
             16ijk
16jkl
             16ijkl
16jl
             16ijl
16k
             16k
             16kl
16kl
161
             161
17
             17
             17ak
17a
17ab
             17abkl
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17abc
17abcd
             12345678abcdijkl
17abi
             17abijkl
17ac
             1478acijk
17acd
             12345678acdijkl
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             124578acijkl
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             123678adijkl
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             17aijk
17ail
             17aijkl
17aj
             17aijk
             17aijkl
17ajl
17al
             17akl
             1478cijk
17c
17cd
             12345678cdijkl
17cl
             124578cijkl
17i
             17ij
             17ijk
17ik
17ikl
             17ijkl
17il
             17ijl
17k
             17k
17kl
             17kl
             18
18
             18ak
18a
18ab
             18abkl
18abd
             1368abdkl
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1368abdikl

18abdi

18abdj 1368abdijkl 18abi 18abikl 18abijkl 18abj 1368adkl 18ad 18adi 1368adikl 1368adijkl 18adj 18ai 18aik 18ail 18aikl 18aj 18aijk 18ajl 18aijkl 18al 18akl 18b 18bl 18bd 1368bdl 18bdi 1368bdil 18bdik 1368bdikl 18bdj 1368bdijl 18bdjk 1368bdijkl 18bdk 1368bdkl 18bi 18bil tetha tetha-bar 18bik 18bikl 18bj 18bijl 18bjk 18bijkl 18bk 18bkl 18d 1368dl 18di 1368dil 18dik 1368dikl 18dj 1368dijl 18djk 1368dijkl 18dk 1368dkl 18i 18i 18ik 18ik 18ikl 18ikl 18il 18il 18 j 18ij 18jk 18ijk 18jkl 18ijkl 18jl 18ijl 18k 18k 18kl 18kl 181 181 1a 1ak 1ab 1abkl 1abc 18abckl 1368abcdkl 1abcd 1368abcdikl 1abcdi 1abcdj 1368abcdijkl 1abci 18abcikl 1abcj 18abcijkl

1abd 16abdkl 1abdi 16abdikl 1abdj 16abdijkl 1abi 1abikl 1abj 1abijkl 18ack 1ac 1acd 1368acdkl 1368acdikl 1acdi 1368acdijkl 1acdj 1aci 18acik 1acil 18acikl 1acj 18acijk 1acjl 18acijkl 18ackl 1acl 16adkl 1ad 1adi 16adikl 1adj 16adijkl 1aik 1ai 1aikl 1ail 1aj 1aijk 1ajl 1aijkl 1akl 1al 1b 1bl 1bc 18bckl 1bcd 1368bcdkl 1bcdi 1368bcdikl 1bcdj 1368bcdijkl 1bci 18bcikl 1bcj 18bcijkl 1bd 16bdl 1bdi 16bdil 1bdik 16bdikl 1bdj 16bdijl 1bdjk 16bdijkl 1bdk 16bdkl 1bil 1bi 1bik 1bikl 1bj 1bijl 1bjk 1bijkl 1bk 1bkl 1c 18ck 1cd 1368cdkl 1cdi 1368cdikl 1cdj 1368cdijkl 18cik 1ci 1cil 18cikl tetha tetha-bar 18cijk 1cj 1cjl 18cijkl

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1cl
              18ckl
1d
              16dl
1di
              16dil
1dik
              16dikl
1dj
              16dijl
1djk
              16dijkl
1dk
              16dkl
1i
              1i
1ik
              1ik
1ikl
              1ikl
1il
              1il
1j
              1ij
1jk
              1ijk
1jkl
              1ijkl
1jl
              1ijl
1k
              1k
              1kl
1kl
11
              11
              ak
a
              abkl
ab
abc
              abckl
abcd
              abcdkl
abcdi
              abcdikl
abcdij
              abcdijkl
              abcikl
abci
abcij
              abcijkl
              abcjkl
abcj
abi
              abikl
              abijkl
abij
              ack
ac
acd
              acdkl
acdi
              acdikl
acdij
              acdijkl
acdj
              acdjkl
aci
              acik
acij
              acijk
acijl
              acijkl
acil
              acikl
              acjk
acj
              acjkl
acjl
              ackl
acl
ad
              adkl
adi
              adikl
adij
              adijkl
              adjkl
adj
              aik
ai
aij
              aijk
aijl
              aijkl
              aikl
ail
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aj
                ajk
                ajkl
ajl
al
                akl
cd
                cdkl
cdi
                cdikl
                cdijkl
cdij
сi
                cik
cij
               cijk
cijl
               cijkl
cil
               cikl
                cjk
сj
cjl
               cjkl
cl
                ckl
i
ij
               ij
ijk
               ijk
ijkl
               ijkl
ik
                ik
ikl
                ikl
il
                il
k
               k
kl
               kl
            599.000000
nr. lines
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Table 3

References

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