Gauss-Seidel's Theorem for Infinite Systems of Linear Equations (II)

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The purpose of this paper is to extend the classical Gauss-Seidel theorem, known for finite linear systems, to infinite one. First of all we need some technical results [4].

1 Vector norms

Let $x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$ be a sequence of real numbers represented in the form of an infinite column vector, and we denote by s the real linear space of these sequences. Let $p \in [1, +\infty)$ be a real number and define $l^p = \{x \in s \mid \sum_{i=0}^{\infty} |x_i|^p \text{ is convergent}\}$. It is well known that l^p is a real linear subspace of s and for every $x \in l^p$ the formula $\|x\|_p = \left(\sum_{i=0}^{\infty} |x_i|^p \right)^{1/p}$ defines a norm on l^p. In this way $(l^p, \|\cdot\|_p)$ is not only a normed linear space, but a Banach space, too. For $p = 1$ and $p = 2$ we reobtain the Banach space l^1 and the Hilbert space l^2, respectively. In l^2 we will consider the standard scalar product given by the formula $(x, y) = \sum_{i=0}^{\infty} x_i y_i$ for every $x, y \in l^2$. For $p, q \in [1, +\infty)$ real numbers from $p < q$ results $l^p \subset l^q$. If s_0 means the linear subspace of convergent sequences to zero then $l^p \subset s_0$ for every $p \in [1, +\infty)$. We also consider the linear subspace $l^\infty = \{x \in s \mid x \text{ is bounded}\}$. For every $x \in l^\infty$ the formula $\|x\|_\infty = \sup_{i \in \mathbb{N}} \{|x_i|\}$ defines a norm on l^∞. In this way $(l^\infty, \|\cdot\|_\infty)$ is not only a normed linear space, but a Banach space, too. We have: $l^1 \subset l^2 \subset s_0 \subset l^\infty \subset s$. All these spaces we will call vector spaces, the elements vectors and the above mentioned norms vector norms [1]. For this paragraph see also [4].

2 Matrix norms

Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be an infinite matrix of real numbers and we denote by M the real linear space of these infinite matrixes. Let $M^1 = \left\{A \in M \mid \sup\limits_{j\in\mathbb{N}} \sum\limits_{i=0}^{\infty} |a_{ij}|\right.$ is finite$\}$. Then M^1 is a real linear subspace of M and for every $A \in M^1$ the formula $\|A\|_1 = \sup\limits_{j\in\mathbb{N}} \sum\limits_{i=0}^{\infty} |a_{ij}|$ defines a norm on M^1 called column norm. In this way $(M^1, \|\cdot\|_1)$ becomes not only a real linear normed space, but a Banach space, too. Let $p \in (1, +\infty)$ be a real number and define

$$M^p = \left\{A \in M \mid \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|^q\right)^{\frac{p}{q}} \text{ is finite}\right\},$$

where q is a real number such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 1. *The space M^p is a real linear subspace of M and for every $A \in M^p$ the formula*

$$\|A\|_p = \left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|^q\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}$$

defines a norm on M^p. The space $(M^p, \|\cdot\|_p)$ is a Banach space.

For $p = 2$ we obtain $M^2 = \left\{A \in M \mid \sum\limits_{i,j=0}^{\infty} a_{ij}^2\right.$ is finite$\}$. If we take on M^2 the scalar product given by the formula $(A, B) = \sum\limits_{i,j=0}^{\infty} a_{ij}b_{ij}$, where $A = (a_{ij})_{i,j\in\mathbb{N}}$ and $B = (b_{ij})_{i,j\in\mathbb{N}}$, then $(M^2, (\cdot,\cdot))$ will be a Hilbert space.

Let $M^\infty = \{A \in M \mid \sup\limits_{i\in\mathbb{N}} \sum\limits_{j=0}^{\infty} |a_{ij}|$ is finite$\}$. Then M^∞ is a real linear subspace of M and for every $A \in M^\infty$ the formula $\|A\|_\infty = \sup\limits_{i\in\mathbb{N}} \sum\limits_{j=0}^{\infty} |a_{ij}|$ defines a norm on M^∞, called row norm. In this way $(M^\infty, \|\cdot\|_\infty)$ becomes not only a normed linear space, but a Banach space, too.

Corollary 1. *If for the matrix* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *we have* $a_{ij} = 0$ *for* $i > n$ *and* $j > n$, $n \in \mathbb{N}$, *then from theorem 1 we reobtain the results in the finite dimensional space* $\mathbb{R}^n$ *[3].*

All these spaces we will call matrix spaces and the above mentioned norms matrix norms. For this paragraph see also [4].

3 The compatibility of the vector and matrix norms

Let $x \in s$ be a sequence of real numbers, and $A = (a_{ij})_{i,j\in\mathbb{N}} \in M$ an infinite matrix of real numbers.

Definition 1. *We will define the product* $A \cdot x$ *if for every* $i \in \mathbb{N}$ *the series* $\sum\limits_{j=0}^{\infty} a_{ij}x_j$ *are convergent. In this case the result vector* $y = A \cdot x$ *is a column vector with components* $y = \begin{pmatrix} \sum\limits_{j=0}^{\infty} a_{0j}x_j \\ \sum\limits_{j=0}^{\infty} a_{1j}x_j \\ \vdots \\ \sum\limits_{j=0}^{\infty} a_{ij}x_j \\ \vdots \end{pmatrix}$

Theorem 2. *For every* $p \in [1, +\infty] = [1, +\infty) \cup \{+\infty\}$ *the vector norm* $\|\cdot\|_p$ *defined on* l^p *is compatible with the matrix norm* $\|\cdot\|_p$ *defined on* M^p, *i.e.* $\|Ax\|_p \leq \|A\|_p \cdot \|x\|_p$ *for every* $x \in l^p$ *and every* $A \in M^p$.

Corollary 2. *If for the matrix* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *we have* $a_{ij} = 0$ *for* $i > n$ *and* $j > n$, $n \in \mathbb{N}$, *then from theorem 2 we reobtain the results in the finite dimensional space* $\mathbb{R}^n$ *[3].*

For this paragraph see also [4].

4 The matrix norm subordinate to a given vector norm

For every $p \in [1, +\infty]$ and for every $x \in l^p$ and $A \in M^p$ we have $\|Ax\|_p \leq \|A\|_p \cdot \|x\|_p$ according to theorem 2. If $x \neq \theta_{l^p}$ (the null element of the vector space l^p)

then $\dfrac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p$ and we can define $\sup\left\{\dfrac{\|Ax\|_p}{\|x\|_p} \mid x \in l^p \setminus \{\theta_{l^p}\}\right\}$. It is known that this formula defines a matrix norm on M^p, which we call the matrix norm subordinate to the vector norm $\|\cdot\|_p$ defined on l^p and we denote by $\|A\|_p^* = \sup\left\{\dfrac{\|Ax\|_p}{\|x\|_p} \mid x \in l^p \setminus \{\theta_{l^p}\}\right\}$. It is immediately that $\|A\|_p^* \leq \|A\|_p$ for every $A \in M^p$.

Theorem 3. *For $p \in \{1, +\infty\}$ we have $\|A\|_p^* = \|A\|_p$.*

Corollary 3. *If for the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ we have $a_{ij} = 0$ for $i > n$ and $j > n$, $n \in \mathbb{N}$, then from theorem 3 we reobtain the results in the finite dimensional space $\mathbb{R}^n$ [3].*

We mention that for the author is unknown how can we calculate for $p \in (1, +\infty)$ the matrix norm subordinate to the vector norm $\|\cdot\|_p$ defined on l^p. For this paragraph see also [4].

The above presented vector and matrix spaces we used to extend the Jacobi's and Gauss-Seidel's methods, known like iterative numerical methods, from finite linear systems to infinite one [5], [6]. In this way we can study the linear stationary processes with infinite but countable number of parameters.

5 Gauss-Seidel's iterative method for infinite systems of linear equations

First let us remember the well known Banach fixed point theorem for Banach spaces:

Theorem 4. *(Banach) Let $(X, \|\cdot\|_X)$ be a Banach space, and Φ a contraction (i.e. there exists a constant $\alpha \in (0,1)$ such that $\|\Phi(x) - \Phi(y)\|_X \leq \alpha \cdot \|x - y\|_X$ for every $x, y \in X$). Then for every $x^0 \in X$ the sequence $(x^k)_{k\in\mathbb{N}}$, generated by the recursion formula $x^{k+1} = \Phi(x^k)$, is convergent and has the limit point $x^* \in X$, which is the unique fixed point of the function Φ in X.*

Let us consider the infinite system of linear equations $Ax = b$, where $A \in M$ and $x, b \in s$.

Definition 2. *For a given $A \in M$ and $b \in s$ we will say that $x^* \in s$ is a solution of the infinite system of linear equations $Ax = b$ if we have $Ax^* = b$.*

This means, that all the series $\sum_{j=0}^{\infty} a_{ij}x_j^*$ are convergent and we have $\sum_{j=0}^{\infty} a_{ij}x_j^* = b_i$ for every $i \in \mathbb{N}$.

Let us suppose, that $a_{ii} \neq 0$ for every $i \in \mathbb{N}$. Then the equation $\sum_{j=0}^{\infty} a_{ij}x_j = b_i$ is equivalent with the equation

$$\begin{aligned} x_i &= \frac{b_i - \sum\limits_{\substack{j=0 \\ j\neq i}}^{\infty} a_{ij}x_j}{a_{ii}}, \quad \text{i.e.} \\ x_i &= -\sum_{\substack{j=0 \\ j\neq i}}^{\infty} \frac{a_{ij}}{a_{ii}}x_j + \frac{b_i}{a_{ii}}. \end{aligned}$$

So the initial system of linear equations $Ax = b$ is equivalent with the following iterative system of linear equations: $x = B \cdot x + c$, where

$$B = \begin{pmatrix} 0 & -\frac{a_{01}}{a_{00}} & \dots & -\frac{a_{0n}}{a_{00}} & \dots \\ -\frac{a_{10}}{a_{11}} & 0 & \dots & -\frac{a_{1n}}{a_{11}} & \dots \\ \vdots & \vdots & & \vdots & \\ -\frac{a_{n0}}{a_{nn}} & -\frac{a_{n1}}{a_{nn}} & \dots & 0 & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} \frac{b_0}{a_{00}} \\ \frac{b_1}{a_{11}} \\ \vdots \\ \frac{b_n}{a_{nn}} \\ \vdots \end{pmatrix}.$$

Let us choose $x^0 \in s$ and we generate the sequence $(x^k)_{k\in\mathbb{N}} \subset s$ by the following iterative formula:

$$\begin{cases} x_0^{k+1} = -\sum\limits_{j=1}^{\infty} \frac{a_{0j}}{a_{00}} x_j^k + \frac{b_0}{a_{00}} \\ x_1^{k+1} = -\frac{a_{10}}{a_{11}} x_0^{k+1} - \sum\limits_{j=2}^{\infty} \frac{a_{1j}}{a_{11}} x_j^k + \frac{b_1}{a_{11}} \\ x_2^{k+1} = -\frac{a_{20}}{a_{22}} x_0^{k+1} - \frac{a_{21}}{a_{22}} x_1^{k+1} - \sum\limits_{j=3}^{\infty} \frac{a_{2j}}{a_{22}} x_j^k + \frac{b_2}{a_{22}} \\ \quad\vdots \\ x_i^{k+1} = -\sum\limits_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \sum\limits_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}} \\ \quad\vdots \end{cases}$$

Consequently from the vector x^k we generate the vector x^{k+1} by the recursion formula $x^{k+1} = B_{GS}x^k + c$. Now we consider the following definition:

Definition 3. *The matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ is l^∞ diagonal dominant if there exists $\lambda \in (0,1)$ such that for every $i \in \mathbb{N}$ we have*

$$\lambda \cdot |a_{ii}| > \sum_{\substack{j=0 \\ j\neq i}}^{\infty} |a_{ij}|.$$

It is immediately that A is l^∞ diagonal dominant if and only if

$$\sup_{i\in\mathbb{N}} \sum_{\substack{j=0 \\ j\neq i}}^{\infty} \left|\frac{a_{ij}}{a_{ii}}\right| < 1.$$

Theorem 5. *If A is l^∞ diagonal dominant then the iterative sequence $(x^k)_{k\in\mathbb{N}}$ is convergent in l^∞ for every $x^0 \in l^\infty$. The limit point $x^* \in l^\infty$ is the unique solution of the linear system $Ax = b$.*

For this result see also [6].

Here we present another proof for theorem 5.

Proof. Let us denote by $\lambda = \sup\limits_{i\in\mathbb{N}} \sum\limits_{\substack{j=0 \\ j\neq i}}^{\infty} \left|\frac{a_{ij}}{a_{ii}}\right| < 1$. We prove by mathematical induction method that $|y_k| \leq \lambda \cdot \|x\|_\infty$ for every $k \in \mathbb{N}$, where $y = B_{GS} \cdot x$. Indeed,

$$\begin{aligned} |y_0| &= \left|-\sum_{j=1}^{\infty} \frac{a_{0j}}{a_{00}} \cdot x_j\right| \leq \sum_{j=1}^{\infty} \left|\frac{a_{0j}}{a_{00}}\right| \cdot |x_j| \leq \\ &\leq \sum_{j=1}^{\infty} \left|\frac{a_{0j}}{a_{00}}\right| \cdot \|x\|_\infty = \left(\sum_{j=1}^{\infty} \left|\frac{a_{0j}}{a_{00}}\right|\right) \cdot \|x\|_\infty \leq \lambda \cdot \|x\|_\infty. \end{aligned}$$

We suppose that $|y_j| \leq \lambda\|x\|_\infty$ for every $j = \overline{0, k-1}$ and we prove that $|y_k| \leq$

$\lambda \cdot \|x\|_\infty$. Indeed,

$$\begin{aligned}
|y_k| &= \left| -\sum_{j=0}^{k-1} \frac{a_{kj}}{a_{kk}} \cdot y_j - \sum_{j=k+1}^{\infty} \frac{a_{kj}}{a_{kk}} \cdot x_j \right| \leq \\
&\leq \sum_{j=0}^{k-1} \left| \frac{a_{kj}}{a_{kk}} \right| \cdot |y_j| + \sum_{j=k+1}^{\infty} \left| \frac{a_{kj}}{a_{kk}} \right| \cdot |x_j| \leq \\
&\leq \sum_{j=0}^{k-1} \left| \frac{a_{kj}}{a_{kk}} \right| \cdot \lambda \cdot \|x\|_\infty + \sum_{j=k+1}^{\infty} \left| \frac{a_{kj}}{a_{kk}} \right| \cdot \|x\|_\infty = \\
&= \left(\sum_{j=0}^{k-1} \left| \frac{a_{kj}}{a_{kk}} \right| \cdot \lambda + \sum_{j=k+1}^{\infty} \left| \frac{a_{kj}}{a_{kk}} \right| \right) \cdot \|x\|_\infty \leq \\
&\leq \left(\sum_{j=0}^{k-1} \left| \frac{a_{kj}}{a_{kk}} \right| + \sum_{j=k+1}^{\infty} \left| \frac{a_{kj}}{a_{kk}} \right| \right) \cdot \|x\|_\infty \leq \lambda \cdot \|x\|_\infty,
\end{aligned}$$

because $\sum_{\substack{j=0 \\ j\neq k}}^{\infty} \left| \frac{a_{kj}}{a_{kk}} \right| \leq \lambda < 1$. Since $|y_k| \leq \lambda \cdot \|x\|_\infty$ for every $k \in \mathbb{N}$ results that

$$\|y\|_\infty = \sup_{k\in\mathbb{N}} \{|y_k|\} \leq \lambda \cdot \|x\|_\infty.$$

This means that

$$\|B_{GS}\|_\infty = \sup_{x\neq\theta_{l^\infty}} \frac{\|B_{GS}x\|_\infty}{\|x\|_\infty} = \sup_{x\neq\theta_{l^\infty}} \frac{\|y\|_\infty}{\|x\|_\infty} \leq \lambda < 1.$$

Now we can apply the Banach fixed point theorem for the iterative function $\Phi : l^\infty \to l^\infty$, $\Phi(x) = B_{GS}x + c$. Indeed, Φ is a contraction, because

$$\|\Phi(x)-\Phi(y)\|_\infty = \|(B_{GS}x+c)-(B_{GS}y+c)\|_\infty = \|B_{GS}(x-y)\|_\infty \leq \|B_{GS}\|_\infty \cdot \|x-y\|_\infty.$$

This means that the sequence $(x^k)_{k\in\mathbb{N}}$ is convergent in l^∞ for every $x^0 \in l^\infty$ and its limit point $x^* \in l^\infty$ is the unique fixed point of Φ in l^∞, i.e. $\Phi(x^*) = x^*$. So $B_{GS}x^* + c = x^*$, which is equivalent with $Ax^* = b$.

□

Corollary 4. *If for the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ we have $a_{ij} = 0$ when $i > n$, $j > n$, and $b_i = 0$ for $i > n$, $n \in \mathbb{N}$, then we reobtain the linear system with finite number of equations and finite number of unknowns. In this way from theorem 5 we obtain the classical Gauss-Seidel's iterative numerical method to solve finite systems of linear equations [2].*

In the next we consider the following definition:

Definition 4. *The matrix* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *is* l^1 *diagonal dominant if there exists* $\lambda \in \left(0, \frac{1}{2}\right)$ *such that for every* $j \in \mathbb{N}$ *we have* $\lambda \cdot |a_{jj}| > \sum\limits_{\substack{i=0\\ i\neq j}}^{\infty} |a_{ij}|$.

It is immediately that A is l^1 diagonal dominant if and only if $\sup_{j\in\mathbb{N}} \sum_{\substack{i=0\\ i\neq j}}^{\infty} \left|\frac{a_{ij}}{a_{jj}}\right| < \frac{1}{2}$.

Theorem 6. *If* A *is* l^1 *diagonal dominant then the iterative sequence* $(x^k)_{k\in\mathbb{N}}$ *is convergent in* l^1 *for every* $x^0 \in l^1$*. The limit point* $x^* \in l^1$ *is the unique solution of the linear system* $Ax = b$*.*

Proof. We have:

$$\begin{aligned}
\|y\|_1 &= \sum_{i=0}^{\infty} |y_i| = \sum_{i=0}^{\infty} \left| -\sum_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} \cdot y_j - \sum_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} \cdot x_j \right| \leq \\
&\leq \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| \cdot |y_j| + \sum_{j=i+1}^{\infty} \left|\frac{a_{ij}}{a_{ii}}\right| \cdot |x_j| \right) \leq \\
&\leq \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| \cdot |x_j| + \sum_{i=j+1}^{\infty} \left|\frac{a_{ij}}{a_{jj}}\right| \cdot |y_j| \right) \leq \\
&\leq \sum_{j=0}^{\infty} \left[\left(\sum_{i=0}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| \right) \cdot |x_j| + \left(\sum_{i=j+1}^{\infty} \left|\frac{a_{ij}}{a_{jj}}\right| \right) \cdot |y_j| \right] \leq \\
&\leq \sum_{j=0}^{\infty} (\lambda \cdot |x_j| + \lambda \cdot |y_j|) = \lambda \cdot \|x\|_1 + \lambda \cdot \|y\|_1.
\end{aligned}$$

Consequently: $\|y\|_1 \leq \lambda \cdot \|x\|_1 + \lambda \cdot \|y\|_1$, which is equivalent with: $\frac{\|y\|_1}{\|x\|_1} \leq \frac{\lambda}{1-\lambda} < 1$.

This means that:

$$\|B_{GS}\|_1 = \sup_{x\neq\theta_{l^1}} \frac{\|B_{GS}x\|_1}{\|x\|_1} = \sup_{x\neq\theta_{l^1}} \frac{\|y\|_1}{\|x\|_1} \leq \frac{\lambda}{1-\lambda} < 1.$$

Now we can apply the Banach fixed point theorem for the iterative function $\Phi : l^1 \to l^1$, $\Phi(x) = B_{GS}x + c$. Indeed, Φ is a contraction, because: $\|\Phi(x) - \Phi(y)\|_1 = \|(B_{GS}x + c) - (B_{GS}y + c)\|_1 = \|B_{GS}(x - y)\|_1 \leq \|B_{GS}\|_1 \cdot \|x - y\|_1$. This means that the sequence $(x^k)_{k\in\mathbb{N}}$ is convergent in l^1 for every $x^0 \in l^1$ and its limit point $x^* \in l^1$ is the unique fixed point of Φ in l^1, i.e. $\Phi(x^*) = x^*$. So $B_{GS}x^* + c = x^*$, which is equivalent with $Ax^* = b$. □

Corollary 5. *If for the matrix* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *we have* $a_{ij} = 0$ *when* $i > n$, $j > n$, *and* $b_i = 0$ *for* $i > n$, $n \in \mathbb{N}$, *then we reobtain the linear system with finite number of* *equations and finite number of unknowns. In this way from theorem 6 we obtain the classical Gauss-Seidel's iterative numerical method to solve finite systems of linear equations [2].*

We mention that for the author is unknown if theorem 6 is true with definition 4 choosing $\lambda \in \left[\frac{1}{2}, 1\right)$.

Using the above presented theorems we can study the linear stationary processes with infinite but countable number of parameters.

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