Closedness of the solution mapping for parametric equilibrium problems

M. Bogdan* and D. Inoan[†]

Abstract

The equilibrium problems are treated as classical inequations to obtain closedness of the solution map defined on the set of parameters.

AMS subject clasification: 49J93D

Key words: Equilibrium problems, Closed set-valued maps, Pseudomonotonicity.

1 Introduction

This work is motivated by two recent papers of Bogdan and Kolumbán [2] and Yu et al. [9]. The first one deals with pseudomonotone operators defined on Sobolev spaces and conclude for the closedness of the solution mapping defined on the set of parameters. The second one contains a result on the behavior of Nash equilibrium points.

We start our exposition from the simple remark: if f is an upper semi-continuous function then the solution set of the inequality $f(x) \ge 0$ is closed.

Let X be a Hausdorff topological space and let $f_n: X \to \mathbb{R}$, $n \in \mathbb{N}$, be given functions. Let us consider, for $n \in \mathbb{N}^*$, the inequations

$$(1)_n f_n(x) \ge 0.$$

Denote by $S_1(1/n, f_n)$ the solutions set of $(1)_n$ and by $S_1(0, f_0)$ the solutions set of

(1)
$$f_0(x) > 0$$
.

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

Let $M=\{f:X\to\mathbb{R}\,|\,\sup_{x\in X}|f(x)|<+\infty\}$ and let ρ be the metric on M given by

$$\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|, f, g \in M.$$

^{*&}quot;Petru Maior" University, ROU - Târgu-Mureș, Romania, e-mail: bgdmarcel@netscape.net †Department of Mathematics, Technical University of Cluj-Napoca, ROU - Cluj-Napoca, Romania, e-mail: Daniela.Inoan@math.utcluj.ro

We shall denote by $f_n \xrightarrow{\rho} f_0$ if $(f_n)_{n \in \mathbb{N}^*}$ converges (uniformly) to f_0 with respect to ρ .

Definition 1. Let M' be a subset of M. We say that $(1)_n$ are stable if the solution mapping $S_1 : \mathbb{N} \times M' \to 2^X$ defined above is closed, i.e. if for each sequence $(x_n)_{n \in \mathbb{N}^*}$, $x_n \in S_1(1/n, f_n)$ with $x_n \to x$, $f_n \xrightarrow{\rho} f_0$ as $n \to +\infty$ one has $x \in S_1(0, f_0)$.

If $M' = uC(X) = \{ f \in M \mid f \text{ is upper semi-continuous on } X \}$ then $(1)_n$ are stable.

Concerning this property there are several results. Lignola and Morgan established convergence of the solutions of generalized parametric variational inequalities (see [6]). Related to Muu's article [7], more recently, Khanh and Luu [4], Li et al. [5] established *closedness* of the solutions function defined on the set of parameters for the same class of problems, namely parametric vector quasivariational inequalities. They have imposed conditions of upper semi-continuity being motivated by restrictions occurring in economical field. However, their hypotheses are too strong in some applications like variational inequalities governed by differential operators (see [2]).

2 Parametric equilibrium problems

Let X be a normed space, D a nonempty subset of X and let $f_n: X \times X \to \mathbb{R}$ be given functions. For $n \in \mathbb{N}^*$ we consider the following parametric equilibrium problem:

$$(2)_n$$
 Find an element $x_n \in D$ such that $f_n(x_n, y) \ge 0, \forall y \in D$.

Denote by $S_2(1/n, f_n)$ the solutions set of $(2)_n$ and by $S_2(0, f_0)$ the solutions set of the "limit" equilibrium problem:

(2) Find an element
$$x_0 \in D$$
 such that $f_0(x_0, y) \ge 0, \forall y \in D$.

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

To compensate for the lack of upper semi-continuity, Brézis ([3]) introduced the notion of topological pseudomonotonicity (which is a kind of conditioned upper semi-continuity) only in the context of variational inequalities (see [8]).

Definition 2. We say that a function $g: X \times X \to \mathbb{R}$ is topologically pseudomonotone (in the sense of Brézis) if for each sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \rightharpoonup x$ in X, $\liminf g(x_n, x) \geq 0$ imply

$$\lim \sup_{n} g(x_n, y) \leq g(x, y), \text{ for all } y \in X.$$

Let $N=\{f:X\times X\to\mathbb{R}\,|\, \sup_{(x,y)\in X\times X}|f(x,y)|<+\infty\}$ and let d be the metric on N given by

$$d(f,g) = \sup_{(x,y) \in X \times X} |f(x,y) - g(x,y)|, \ f, g \in N.$$

We shall denote by $f_n \xrightarrow{d} f_0$ if $(f_n)_{n \in \mathbb{N}^*}$ converges (uniformly) to f_0 with respect to d.

Denote $pC(X,X) = \{ f \in N \mid f \text{ is topologically pseudomonotone } \}.$

Proposition 1. Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence such that $f_n \in pC(X,X)$. If $f_n \stackrel{d}{\longrightarrow} f_0$ then $f_0 \in pC(X,X)$.

Proof. Let $\varepsilon > 0$. Let $(x_n)_{n \in \mathbb{N}}$ with $x_n \to x$ in X, and $\liminf_n f_0(x_n, x) \geq 0$. For n large we gain $\liminf_n f_n(x_n, x) \geq 0$. Since f_n are in pC(X, X) one has

$$\limsup_{n} f_n(x_n, y) \le f_n(x, y), \text{ for all } y \in X.$$

Thus

$$\lim \sup_{n} f_0(x_n, y) \le f_n(x, y) < f_0(x, y) + \varepsilon, \text{ for all } y \in X.$$

Letting $\varepsilon \to 0$ the conclusion follows.

Definition 3. Let N' be a subset of N. We say that $(2)_n$ are stable if the solution mapping $S_2 : \mathbb{N} \times N' \to 2^X$ defined above is closed, i.e. if for each sequence $(x_n)_{n \in \mathbb{N}^*}$, $x_n \in S_2(1/n, f_n)$ with $x_n \to x_0$, $f_n \xrightarrow{d} f_0$ as $n \to +\infty$ one has $x_0 \in S_2(0, f_0)$.

Proposition 2. If N' = pC(X, X) then $(2)_n$ are stable.

Proof. The conclusion follows by the following inequalities

$$0 \leq \liminf_{n} f_n(x_n, y) \leq \limsup_{n} f_0(x_n, y) + \liminf_{n} \left(f_n(x_n, y) - f_0(x_n, y) \right)$$
$$= \limsup_{n} f_0(x_n, y) \leq f_0(x, y), \forall y \in D.$$

3 Parametric Domains

To this point the feasible set has been "constant". Corresponding to $(1)_n$ for $n \in \mathbb{N}^*$ we shall consider the parametric inequations:

$$(1)'_n$$
 find $x \in D_n$ such that $f_n(x) \ge 0$,

where $D_n \subseteq X$ are nonempty and closed.

Denote by $S'_1(1/n, f_n)$ the solutions set of $(1)'_n$ and by $S'_1(0, f_0)$ the solutions set of

(1)' find
$$x \in D_0$$
 such that $f_0(x) \ge 0, x \in D_0$.

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$. Clearly $S'_1(1/n, f_n) = f_n^{-1}([0, +\infty)) \cap D_n$ and $S'_1(0, f_0) = f_0^{-1}([0, +\infty)) \cap D_0$. If $\emptyset \neq \cap_{n \in \mathbb{N}^*} D_n \subseteq D_0$ we can easily conclude that $(1)'_n$ are stable in the sense of Definition 1.

Corresponding to $(2)_n$ we shall consider the parametric equilibrium problems:

 $(2)'_n$ Find an element $x_n \in D_n$ such that $f_n(x_n, y) \ge 0, \forall y \in D_n$.

Denote by $S'_2(1/n, f_n)$ the solutions set of $(2)'_n$ and by $S'_2(0, f_0)$ the solutions set of the "limit" equilibrium problem:

(2)' Find an element $x_0 \in D_0$ such that $f_0(x_0, y) \ge 0, \forall y \in D_0$.

Suppose that solutions sets are not empty for all $n \in \mathbb{N}$.

Definition 4. We say that $(2)'_n$ are stable if the solution map $S'_2: \mathbb{N} \times pC(X, X) \to 2^X$ is closed, i.e. if $x_n \in S'_2(1/n, f_n)$, $x_n \rightharpoonup x_0$, and $f_n \stackrel{d}{\longrightarrow} f_0$ in pC(X, X) as $n \to +\infty$, then $x_0 \in S'_2(0, f_0)$.

We have the following statement.

Theorem 1. Let X be a normed space. Suppose that D_n are weakly closed for all $n \in \mathbb{N}^*$ and $\emptyset \neq \cap_{n \in \mathbb{N}^*} D_n = D_0$.

Then $(2)'_n$ are stable in the sense of Definition 4.

Proof. Let $x_n \in D_n$ be such that $(2)'_n$ is valid. Let $x_n \rightharpoonup x_0$ in X thus we have $x_0 \in D_n$, so $\liminf_n f_n(x_n, x_0) \ge 0$. Since $f_n \stackrel{d}{\longrightarrow} f_0$ it follows

$$\liminf_{n} f_0(x_n, x_0) \ge 0$$

therefore, by topological pseudomonotonicity of f_0 one has

$$\lim \sup_{n} f_0(x_n, y) \le f_0(x_0, y), \text{ for all } y \in D_0.$$

Finally,

$$0 \leq \liminf_{n} f_n(x_n, y) \leq \limsup_{n} f_0(x_n, y) + \liminf_{n} \left(f_n(x_n, y) - f_0(x_n, y) \right) \leq \lim_{n} \sup_{n} f_0(x_n, y) \leq f_0(x_0, y), \forall y \in D_0,$$

which completes the proof.

One question rises. What is the condition for the parametric domains in order to have the stability of $(2)'_n$? We are not naive to consider condition $\emptyset \neq \cap_{n \in \mathbb{N}^*} D_n = D_0$ a proper one, especially in applications. In [2] we used Mosco's convergence.

4 Application to hemivariational inequalities

Let us denote by $pC_0(X, X) = \{ f \in pC(X, X) : f(x, x) = 0, x \in X \}$. First, we establish that $pC_0(X, X)$ is stable with respect to addition. The proof is similar to Proposition 2.4 in [8].

Proposition 3. If $f, g \in pC_0(X, X)$ then $f + g \in pC_0(X, X)$.

Proof. We claim that $\liminf_n f(x_n, x) \geq 0$ and $\liminf_n g(x_n, x) \geq 0$. Otherwise $\liminf_n g(x_n, x) = -\varepsilon < 0$ and, by passing to a subsequence $\lim_n g(x_n, x) = -\varepsilon$. Therefore

$$\liminf_{n} f(x_n, x) = \liminf_{n} ((f+g)(x_n, x) - g(x_n, x))$$

$$= \liminf_{n} (f+g)(x_n, x) - \lim_{n} g(x_n, x) \ge 0 + \varepsilon.$$

Since f is topologically pseudomonotone one has

$$\lim \sup_{n} f(x_n, y) \le f(x, y), \text{ for all } y \in X.$$

Now, put x = y to reach the contradiction as follows

$$\varepsilon \le \liminf_{n} f(x_n, x) \le \limsup_{n} f(x_n, x) \le f(x, x) = 0.$$

The conclusion is obtained from the super-additivity of \liminf and sub-additivity of \limsup .

Let Ω be a bounded open set of \mathbb{R}^N with Lipschitz boundary. Let us consider a function $l: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that $l(x; \cdot, \cdot)$ is upper semi-continuous for a.e. $x \in \Omega$ and $l(\cdot; y, z)$ is measurable for all $y, z \in \mathbb{R}$. We say that l satisfies the growth condition if there exist $h_1, h_2 \in \mathbb{R}_+$ such that

$$|l(x; y, z)| \le (h_1 + h_2|y|)|z|$$
, a.e. $x \in \Omega, \forall y, z \in \mathbb{R}$.

Lemma 1. ([2]) If l satisfies the conditions above, then the function defined on $H^1(\Omega) \times H^1(\Omega)$ by

$$(u,w) \mapsto \int_{\Omega} l(x;u(x),w(x)) dx$$

is weakly upper semi-continuous.

For $n \in \mathbb{N}^*$ we consider the following variational inequality problem: $(VI)_n$ Find an element $u_n \in K$ such that

$$\int_{\Omega} A_n(x, u_n(x), \nabla u_n(x)) \cdot (\nabla v(x) - \nabla u_n(x)) dx +$$

$$+ \int_{\Omega} a^n(x, u_n(x), \nabla u_n(x)) (v(x) - u_n(x)) dx +$$

$$+ \int_{\Omega} l_n(x; u_n(x), v(x) - u_n(x)) dx \ge 0, \forall v \in K,$$

with $A_n = (a_1^n, \dots, a_N^n)$, a^n , and l_n given functions, and K a closed, convex, nonempty subset of the Sobolev space $H^1(\Omega)$.

For a fixed $n \in \mathbb{N}$, we consider the operator

$$\mathcal{A}_n: H^1(\Omega) \to (H^1(\Omega))^*, \qquad \mathcal{A}_n(u) = \mathcal{B}_n(u, u),$$

where

$$\mathcal{B}_n(u,v)(w) = \int_{\Omega} \left\{ \sum_{i=1}^N a_i^n(x,u(x),\nabla u(x)) \cdot \partial_i w(x) \right\} dx + \int_{\Omega} a^n(x,u(x),\nabla u(x)) w(x) dx.$$

The operators A_n are topologically pseudomonotone if satisfies some reasonable conditions (see [8], page 76; [2]).

 $(VI)_n$ can be written as

find
$$u_n \in K$$
 such that $f_n(u_n, v) + L_n(u_n, v) \ge 0, \forall v \in K$,

where
$$f_n(u,v) = \langle \mathcal{A}_n(u), v - u \rangle$$
 and $L_n(u,v) = \int_{\Omega} l_n(x; u(x), v(x) - u(x)) dx$.
Finally, we can apply Proposition 2 and Proposition 3 to obtain a result on the

stability of $(VI)_n$.

References

- [1] Anh, L. Q., and Khanh, P. Q.: Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces, Journal of Global Optimization, original paper, DOI 10.1007/s10898-006-9062-8 (2006).
- [2] Bogdan, M., and Kolumbán, J.: On Nonlinear Parametric Variational Inequalities, Nonlinear Analysis, Vol. 67, No. 7, pp. 2272 - 2282, 2007.
- [3] Brézis, H.: Équations et inéquations non linéaires en dualité, Annales de l'Institut Fourier (Grenoble) 18(1)(1965), pp. 115 - 175.
- [4] Khanh, P. Q., and Luu, L. M.: Upper Semicontinuity of the Solution Set to Parametric Vector Quasivariational Inequalities, Journal of Global Optimization, 32(2005), 569 - 580.
- [5] Li, S. J., Chen, G. Y., and Teo, K. L.: On the Stability of Generalized Vector Quasivariational Inequality Problems, Journal of Optimization Theory and Applications, vol. 107, no. 1, pp. 35 - 50, 2000.
- [6] Lignola, M. B., and Morgan, J.: Generalized Variational Inequalities with Pseudomonotone Operators Under Perturbations, Journal of Optimization Theory and Applications, Vol. 101, No. 1, pp. 213 - 220, 1999.
- [7] Muu, L. D.: Stability property of a class of variational inequalities, Mathematische Operationforschung und Statistik, Serie Optimization 15, pp. 347 - 351, 1984.
- [8] Showalter, R. E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, American Mathematical Society, vol. 49, 1997.
- [9] Yu, J., Yang, H., and Yu, C.: Well-posed Ky Fan's point, quasi-variational inequality and Nash equlibrium problems, Nonlinear Analysis, Vol. 66, pp. 777 - 790, 2007.