

# Product of binary relations

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Let  $I \neq \emptyset$  be a set of indices,  $(X_i)_{i \in I}$  and  $(Y_i)_{i \in I}$  two families of sets. Let us consider the binary relations  $\rho_i = (X_i, Y_i, G_i)$ ,  $i \in I$ , where  $G_i$  is the graph of  $\rho_i$ . At the same time we take the cartesian products  $X = \prod X_i$  and  $Y = \prod Y_i$ . We mention that, when  $I$  is a finite set, then there exist the cartesian products  $X$  and  $Y$ , but when  $I$  is an infinite set, then we need the axiom of choice to the existence of the cartesian products  $X$  and  $Y$ .

**Definition 1.** We will say that the element  $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$  is in the relation  $\rho$  with the element  $y = (y_i)_{i \in I} \in \prod_{i \in I} Y_i$ , denoted by  $x\rho y$ , if for all  $i \in I$  the elements  $x_i \in X_i$  are in the relation  $\rho_i$  with  $y_i \in Y_i$ , i.e.  $x_i\rho_i y_i$  for every  $i \in I$ .

The relation  $\rho$  we will call the product of the binary relations  $(\rho_i)_{i \in I}$  and we will denote by  $\rho = \prod_{i \in I} \rho_i$ . We can observe that the graph of the relation  $\rho$  is the set  $G \subset X \times Y = \left( \prod_{i \in I} X_i \right) \times \left( \prod_{i \in I} Y_i \right)$  given by  $G = \{(x, y) / x = (x_i)_{i \in I} \in \prod_{i \in I} X_i = X, y = (y_i)_{i \in I} \in \prod_{i \in I} Y_i = Y \text{ and } x_i\rho_i y_i \text{ for every } i \in I\}$ .

So the product relation is  $\rho = (X, Y, G)$ .

**Proposition 1.** The element  $(x, y) \in G$  if and only if  $((x_i, y_i))_{i \in I} \in \prod_{i \in I} G_i$ .

*Proof.* The element  $(x, y) \in G$  if and only if  $x_i\rho_i y_i$  for all  $i \in I$ , i.e.  $(x_i, y_i) \in G_i$  for every  $i \in I$ , which means that  $((x_i, y_i))_{i \in I} \in \prod_{i \in I} G_i$ .  $\square$

Let  $A \subset X$  be a subset of  $X$ .

**Definition 2.** The set  $\rho(A) = \text{Im}(A) = \{y \in Y / \text{there exist } x \in A \text{ such that } x\rho y\}$  we will call the direct image of the set  $A$  by the relation  $\rho$ .

**Proposition 2.** The following assertions are true:

1.  $\rho(\Phi) = \text{Im}(\Phi) = \Phi$ ;
2. If  $A \subset X$  and  $A' \subset X$  are two subsets of  $X$ , then:
  - a)  $A \subset A'$  implies  $\rho(A) \subseteq \rho(A')$ ;

- b)  $\rho(A \cup A') = \rho(A) \cup \rho(A')$ ;  
c)  $\rho(A \cap A') \subseteq \rho(A) \cap \rho(A')$ .

*Proof.* The proof is standard. □

For every  $i \in I$  let us consider the subsets  $A_i \subset X_i$  and  $A = \prod_{i \in I} A_i \subseteq \prod_{i \in I} X_i = X$ .

**Proposition 3.** *We have  $\rho(A) = \prod_{i \in I} \rho_i(A_i)$ .*

*Proof.* The element  $y = (y_i)_{i \in I} \in \rho(A)$  iff there exists  $x = (x_i)_{i \in I} \in \prod_{i \in I} A_i$  such that  $x\rho y$ , i.e. for every  $y_i$  there exist  $x_i \in A_i$  such that  $x_i \rho_i y_i$ , which means that  $y_i \in \rho_i(A_i)$  for every  $i \in I$ , so  $y = (y_i)_{i \in I} \in \prod_{i \in I} \rho_i(A_i)$ . □

Let  $B \subset Y$  be a subset of  $Y$ .

**Definition 3.** *The set  $\rho^{-1}(B) = \{x \in X \mid \rho(x) \in B\}$  we will call the inverse image of the set  $B$  by the relation  $\rho$ .*

**Proposition 4.** *If  $B \subset Y$  and  $B' \subset Y$  are two subsets of  $Y$ , then the following assertions are true:*

1.  $B \subseteq B'$  implies  $\rho^{-1}(B) \subseteq \rho^{-1}(B')$ ;
2.  $\rho^{-1}(B) \cup \rho^{-1}(B') = \rho^{-1}(B \cup B')$ ;
3.  $\rho^{-1}(B \cap B') = \rho^{-1}(B) \cap \rho^{-1}(B')$ ;
4.  $\rho^{-1}(Y) = X$ .

*Proof.* The proof is standard. □

For every  $i \in I$  let us consider the subsets  $B_i \subset Y_i$  and  $B = \prod_{i \in I} B_i \subset \prod_{i \in I} Y_i = Y$ .

**Proposition 5.** *We have  $\rho^{-1}(B) = \prod_{i \in I} \rho_i^{-1}(B_i)$ .*

*Proof.* The element  $x = (x_i)_{i \in I} \in \rho^{-1}(B)$ , iff  $\rho(x) \in B = \prod_{i \in I} B_i$ , i.e.  $\rho_i(x_i) \in B_i$  for every  $i \in I$ , which means that  $x_i \in \rho_i^{-1}(B_i)$  for every  $i \in I$ , therefore  $x = (x_i)_{i \in I} \in \prod_{i \in I} \rho_i^{-1}(B_i)$ . □

Let us consider the sets  $(Z_i)_{i \in I}$  and the binary relations  $\gamma_i = (Y_i, Z_i, H_i)$ ,  $i \in I$ , where  $H_i$  are the graph of the binary relations  $\gamma_i$ . Let us denote by  $\gamma$  the product of the binary relations  $(\gamma_i)_{i \in I}$ , i.e.  $\gamma = \prod_{i \in I} \gamma_i$ , with the graph  $H = \{(y, z) \mid y = (y_i)_{i \in I} \in \prod_{i \in I} Y_i = Y, z = (z_i)_{i \in I} \in \prod_{i \in I} Z_i = Z \text{ and } y_i \gamma_i z_i \text{ for every } i \in I\}$ . So the product relation is  $\gamma = (Y, Z, H)$ .

**Definition 4.** We will call the composition of the relation  $\rho = (X, Y, G)$  with the relation  $\gamma = (Y, Z, H)$  the following relation:  $\gamma \circ \rho = (X, Z, H \circ G)$ , where  $H \circ G = \{(x, z) \mid \text{there exists } y \in Y \text{ such that } (x, y) \in G \text{ and } (y, z) \in H\}$ . In this case we will say that the element  $x \in X$  is in the relation  $\gamma \circ \rho$  with the element  $z \in Z$ , and we denote by  $x \gamma \circ \rho z$ .

**Proposition 6.** The element  $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i = X$  is in the relation  $\gamma \circ \rho$  with the element  $z = (z_i)_{i \in I} \in \prod_{i \in I} Z_i = Z$  iff for every  $i \in I$  the element  $x_i \in X_i$  is in the relation  $\gamma_i \circ \rho_i$  with the element  $z_i \in Z_i$ .

*Proof.* We have  $x \gamma \circ \rho z$  if and only if there exists  $y \in Y$  such that  $x \rho y$  and  $y \gamma z$ . But  $x \rho y$  and  $y \gamma z$  means that for every  $i \in I$   $x_i \rho_i y_i$  and  $y_i \gamma_i z_i$ , i.e.  $x_i \gamma_i \circ \rho_i z_i$  for all  $i \in I$ .  $\square$

Let us consider the sets  $(U_i)_{i \in I}$  and the binary relations  $\sigma_i = (Z_i, U_i, J_i)$ ,  $i \in I$ , where  $J_i$  is the graph of the binary relation  $\sigma_i$ . Let us denote by  $\sigma$  the product of the binary relations  $(\sigma_i)_{i \in I}$ . So  $\sigma = \prod_{i \in I} \sigma_i$ , i.e.  $\sigma = (Z, U, J)$ , where  $Z = \prod_{i \in I} Z_i$ ,  $U = \prod_{i \in I} U_i$  and  $J = \{(z, u) \mid z \sigma u, z \in Z \text{ and } u \in U\}$ .

**Proposition 7.** Let  $\rho = (X, Y, G)$ ,  $\gamma = (Y, Z, H)$ ,  $\sigma = (Z, U, J)$  be three relations. Then  $\sigma \circ (\gamma \circ \rho) = (\sigma \circ \gamma) \circ \rho$ .

*Proof.* The proof is standard.  $\square$

## References

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