

# The operatorial form of the overdetermined infinite linear systems

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Proceedings of the European Integration between Tradition and  
Modernity

"Petru Maior" University of Târgu-Mureş

Oktober, 22-23, 2009

Let  $X_1$  and  $X_2$  be two real or complex Hilbert spaces, respectively. We consider the linear and continuous operator  $T : X_1 \rightarrow X_2$ , where  $T^* : X_2 \rightarrow X_1$  is the adjoint operator of  $T$ . Let us take the equation  $T(x) = b$ , where  $x \in X_1$  is the unknown and  $b \in X_2$  is a fixed element.

**Theorem 1.** *If  $x^* \in X_1$  verifies the condition  $(T(x^*) - b) \in \text{Ker}(T^*)$  then  $\|T(x^*) - b\| \leq \|T(x) - b\|$  for all  $x \in X_1$ .*

*Proof.* We have the following:

$$\begin{aligned} \|b - T(x)\|^2 &= \|b - T(x^*) + T(x^* - x)\|^2 = \\ &= (b - T(x^*) + T(x^* - x), b - T(x^*) + T(x^* - x)) = \\ &= (b - T(x^*), b - T(x^*)) + (b - T(x^*), T(x^* - x)) + \\ &\quad + (T(x^* - x), b - T(x^*)) + (T(x^* - x), T(x^* - x)) = \\ &= \|b - T(x^*)\|^2 + (T^*(b - T(x^*)), x^* - x) + \\ &\quad (x^* - x, T^*(b - T(x^*))) + \|T(x^* - x)\|^2 = \\ &= \|b - T(x^*)\|^2 + \|T(x^* - x)\|^2 \geq \|b - T(x^*)\|^2. \end{aligned}$$

□

Next we give some applications.

**Application 1.** We consider the real, finite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \dots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

where  $m > n$  and  $a_{ij}, b_i \in \mathbb{R}$  for all  $i = \overline{0, m}$  and  $j = \overline{1, n}$ . Let  $A = (a_{ij})_{\substack{i=\overline{0, m} \\ j=\overline{1, n}}}$  be the matrix of the real linear system and  $b = (b_i)_{i=\overline{0, m}}$  is the constant term. Then we obtain the following linear and continuous operator  $T : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ , and  $T(x) = A \cdot x$ , for every  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ . So the adjoint operator  $T^*$  has the matrix  $A^T$ , which is the transpose of the matrix  $A$ . Using our theorem, from the condition  $(T(x^*) - b) \in \text{Ker}(T^*)$  we obtain  $T^*(T(x^*) - b) = 0$ , i.e.  $T^*(T(x^*)) = T^*(b)$ , which has the equivalent matrix form  $A^T \cdot A \cdot x^* = A^T \cdot b$ . So from our theorem we reobtain the following well known result: if  $A^T \cdot A \cdot x^* = A^T \cdot b$  then  $\|A \cdot x^* - b\| \leq \|A \cdot x - b\|$  for all  $x \in \mathbb{R}^n$  (see [1], [2] or [3]).

**Application 2.** We consider the real, infinite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \dots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ \vdots \end{cases}$$

where  $a_{ij}, b_i \in \mathbb{R}$  for all  $i \in \mathbb{N}$  and  $j = \overline{1, n}$ . Let  $a_j = (a_{ij})_{i \in \mathbb{N}} \in l^2(\mathbb{R})$  and  $b = (b_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ , so  $A = (a_1 a_2 \dots a_n)$  is the matrix of the infinite linear system and  $b$  is the constant term. Then we obtain the following linear and continuous operator  $T : \mathbb{R}^n \rightarrow l^2(\mathbb{R})$ ,  $T(x) = A \cdot x$ , for every  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ . So the adjoint operator  $T^*$  has the matrix  $A^T$ , which is the transpose of the matrix  $A$ . Using our theorem, from the condition  $(T(x^*) - b) \in \text{Ker}(T^*)$  we obtain  $T^*(T(x^*) - b) = 0$ , i.e.  $T^*(T(x^*)) = T^*(b)$ , which has the equivalent matrix form  $A^T \cdot A \cdot x^* = A^T \cdot b$ . So from our theorem we reobtain the following result: if  $A^T \cdot A \cdot x^* = A^T \cdot b$  then  $\|A \cdot x^* - b\| \leq \|A \cdot b - b\|$  for all  $x \in \mathbb{R}^n$  (see [4]).

**Application 3.** We consider the complex, finite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \dots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

where  $m > n$  and  $a_{ij}, b_i \in \mathbb{C}$  for all  $i = \overline{0, m}$  and  $j = \overline{1, n}$ . Let  $A = (a_{ij})_{\substack{i=\overline{0, m} \\ j=\overline{1, n}}}$  be the matrix of the complex linear system and  $b = (b_i)_{i=\overline{0, m}}$  is the constant term. Then we obtain the following linear and continuous operator  $T : \mathbb{C}^n \rightarrow \mathbb{C}^{m+1}$ ,  $T(x) = A \cdot x$ , for every  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ . So the adjoint operator  $T^*$  has the matrix  $\overline{A}^T$ , which is the transpose of the matrix  $A$  and taking the complex conjugate for all elements of  $A^T$ . Using our theorem, from the condition  $(T(x^*) - b) \in \text{Ker}(T^*)$  we obtain  $T^*(T(x^*) - b) = 0$ , i.e.  $T^*(T(x^*)) = T^*(b)$ , which has the equivalent matrix form from  $\overline{A}^T \cdot A \cdot x^* = \overline{A}^T \cdot b$ . It is immediately that this last relation is the same with  $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$ . So from our theorem we reobtain the following well known result: if  $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$  then  $\|A \cdot x^* - b\| \leq \|A \cdot x - b\|$  for all  $x \in \mathbb{C}^n$  (see [1], [2] or [3]).

**Application 4.** We consider the complex, finite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \dots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ \vdots \end{cases}$$

where  $a_{ij}, b_i \in \mathbb{C}$  for all  $i \in \mathbb{N}$  and  $j = \overline{1, n}$ . Let  $a_j = (a_{ij})_{i \in \mathbb{N}} \in l^2(\mathbb{C})$  and  $b = (b_i)_{i \in \mathbb{N}} \in l^2(\mathbb{C})$ , so  $A = (a_1 a_2 \dots a_n)$  is the matrix of the infinite linear system and  $b$  is the constant term. Then we obtain the following linear and continuous operator  $T : \mathbb{C}^n \rightarrow l^2(\mathbb{C})$ ,  $T(x) = A \cdot x$ , for every  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ . So the adjoint operator  $T^*$  has the matrix  $\overline{A}^T$ , which is the transpose of the matrix  $A$  and taking the complex conjugate for all elements of  $A^T$ . Using our theorem, from the condition  $(T(x^*) - b) \in \text{Ker}(T^*)$  we obtain  $T^*(T(x^*) - b) = 0$ , i.e.  $T^*(T(x^*)) = T^*(b)$ , which has the equivalent matrix from  $\overline{A}^T \cdot A \cdot x^* = \overline{A}^T \cdot b$ . It is immediately that this last relation is the same with  $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$ . So from our theorem we obtain the following result: if  $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$  then  $\|A \cdot x^* - b\| \leq \|A \cdot x - b\|$  for all  $x \in \mathbb{C}^N$ . (see [5]).

## References

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