

GENERATED MOSCO CONVERGENCE FOR SETS

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Abstract

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. We emphasize some formalisms in normed spaces that generate convergent sequences of sets. A different approach for a kind of restriction sets that proves Mosco convergence is given.

Key words: Parametric problems, Equilibrium problems, Variational inequalities, Mosco convergence, Topological pseudomonotonicity.

There are numerous problems in nonlinear analysis, like scalar and/or vector equilibrium problems, scalar and/or vector variational inequality problems, where parametric domains occur (see [2, 5, 6]).

Let X be a normed space.

Let $A, A_n, n \in \mathbb{N}$ be nonempty sets in X . We shall use the following notations:

$$\liminf A_n = \{x \in X \mid \exists(x_n), x_n \in A_n, \forall n \in \mathbb{N}, x_n \rightarrow x\};$$

$$\limsup A_n = \{x \in X \mid \exists(n_k), \exists(x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \rightarrow x\};$$

$$w - \limsup A_n = \{x \in X \mid \exists(n_k), \exists(x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \xrightarrow{w} x\},$$

where w denotes the weak convergence in X .

Definition 1.

(PK) The sequence $(A_n)_{n \in \mathbb{N}}$ is said to converges in the Painlevé-Kuratowski sense to A and notes $A_n \xrightarrow{PK} A$ if

$$\limsup A_n \subseteq A \subseteq \liminf A_n;$$

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(M) The sequence $(A_n)_{n \in \mathbb{N}}$ is said to converges in the Mosco sense to A and notes $A_n \xrightarrow{M} A$ if

$$w - \limsup A_n \subseteq A \subseteq \liminf A_n.$$

It is clear that if $A_n \xrightarrow{M} A$ then $A_n \xrightarrow{PK} A$.

It is easy to provide examples in \mathbb{R} . Let $n \geq 1$, $A_n = \{-1/n, n\}$ for n odd, and $A_n = \{1/n\}$ for n even. Then $A_n \xrightarrow{PK} A$, where $A = \{0\}$.

The parametrization concept can be done for the discrete mode, i.e. the parameter is $n \in \mathbb{N}$ or a continuous mode when the parameter is considered in a topological (or metric) space. It is the last case when the parameter is the time t in an interval $I \subseteq \mathbb{R}$ (see [3]). A slight generalization for sets convergence has been used in this case. Let (X, σ) be a Hausdorff topological spaces, let P (the set of parameters) be also a Hausdorff topological space and let $p_0 \in P$ be fixed. Along with the topology σ , we also consider a stronger topology τ on X . If X is a normed space and $\sigma = \tau =$ norm topology, $D_p \xrightarrow{M} D_{p_0}$ amounts to saying that the sets D_p converge to D_{p_0} in the Painlevé-Kuratowski sense as $p \rightarrow p_0$. If X is a normed space and σ is chosen as the weak topology and τ as the norm topology, then obviously, $D_p \xrightarrow{M} D_{p_0}$ implies $D_p \xrightarrow{PK} D_{p_0}$ as $p \rightarrow p_0$. It is useful the following form.

Definition 2. Let D_p be subsets of X for all $p \in P$. The sets D_p converge to D_{p_0} (and write $D_p \xrightarrow{M} D_{p_0}$) as $p \rightarrow p_0$ if:

- (a) for every net $(a_{p_i})_{i \in I}$ with $a_{p_i} \in D_{p_i}$, $p_i \rightarrow p_0$ and $a_{p_i} \xrightarrow{\sigma} a$ imply $a \in D_{p_0}$;
- (b) for every $a \in D_{p_0}$, there exist $a_p \in D_p$ such that $a_p \xrightarrow{\tau} a$ as $p \rightarrow p_0$.

In [1] is described a concrete situation in order to generate Mosco convergence. For a reflexive Banach space X , a set-valued mapping $D : X \rightarrow 2^X$ is defined by

$$D(x) = D_0 + d(x),$$

where D_0 is a closed convex nonempty subset of X and $d : X \rightarrow X$ is a compact map (i.e. a weakly-strongly continuous map).

Proposition 1. ([1], Proposition 1) For any sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \xrightarrow{w} x$ in X , then $D(x_n) \xrightarrow{M} D(x)$.

A partial result from [1], Theorem 1 can be deduced from our result Theorem 1 in [4].

Are there other formalisms that generate Mosco convergence for sets? Certainly, among them see Lemma 1.4, 1.6 in [6].

This type of convergence was introduced by Umberto Mosco in [6]. His examples were the following:

Example 1. Let $X = l^2 = \{y = (y_k)_{k \in \mathbb{N}} \mid \sum_{k=1}^{\infty} |y_k|^2 < +\infty\}$. Let B the unit ball in X with respect to the usual norm $\|y\| = (\sum_{k=1}^{\infty} |y_k|^2)^{1/2}$. Denote by $S_0 = (2B) \cap \{y = (y_k)_{k \in \mathbb{N}} \mid 0 \leq y_k \leq 1, \forall k \in \mathbb{N}\}$ and $S_n = (2B) \cap \{y = (y_k)_{k \in \mathbb{N}} \mid 0 \leq y_k \leq 1 + k/n, \forall k \in \mathbb{N}\}$. One has $S_n \xrightarrow{M} S_0$.

Let $E = \{e_k \mid k \in \mathbb{N}\}$ be the canonical basis. Denote by $C_0 = \text{cl conv}\{e_1, e_2, \dots, e_k, \dots\}$ and $C_n = \text{cl conv}\{(1+1/n)e_1, (1+2/n)e_2, \dots, (1+k/n)e_k, \dots\}$. One has $C_n \xrightarrow{M} C_0$.

Now, conform Proposition 1.2.1 in [2], for $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}$, one has

- $\limsup(A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n$
- $\liminf(A_n \cap B_n) \subseteq \liminf A_n \cap \liminf B_n$.

The question rises: if $A_n \xrightarrow{PK} A$ and $B_n \xrightarrow{PK} B$ when $A_n \cap B_n \xrightarrow{PK} A \cap B$? The same: if $A_n \xrightarrow{M} A$ and $B_n \xrightarrow{M} B$ when $A_n \cap B_n \xrightarrow{M} A \cap B$?

There is an answer given in [7], Proposition 27 along with the constrained condition $X = \mathbb{R}_+(A - B)$.

In [3] the following kind of sets occurs:

$$K(t) = \{x(t) \in \mathbb{R}^m : \lambda(t) \leq x(t) \leq \mu(t), M(t) \cdot x(t) = b(t)\},$$

where $\lambda, \mu \in C([0, T], \mathbb{R}^m)$ with $\lambda < \mu$ and $b \in C([0, T], \mathbb{R}^l_+)$ are vector-functions and $M \in C([0, T], \mathbb{R}^{l \times m}_+)$ is a matrix-function.

Proposition 2. ([3]) *Let $(t_n)_{n \in \mathbb{N}} \subset [0, T]$ be a sequence such that $t_n \rightarrow t$. Then*

$$K(t_n) \xrightarrow{PK} K(t), \text{ as } n \rightarrow \infty.$$

We shall consider a more simple case but a different approach. Define the closed, convex sets

$$A(t) = \{x(t) \in \mathbb{R}^m \mid \lambda(t) \leq x(t)\}.$$

Since the product sets enjoys the following properties:

- $\limsup \prod_{k=1}^m A_n^k \subseteq \prod_{k=1}^m \limsup A_n^k$;
- $\liminf \prod_{k=1}^m A_n^k = \prod_{k=1}^m \liminf A_n^k$,

we can conclude that $A(t_n) \xrightarrow{PK} A(t)$, as $n \rightarrow \infty$ if this is proved for $m = 1$.

Let $x(t) \in A(t)$. Define

$$x(t_n) := x(t) - \lambda(t) + \lambda(t_n).$$

We have $x(t_n) \in A(t_n)$ and by the continuity of λ we get $\lim_{n \rightarrow \infty} x(t_n) = x(t)$, that is $A(t) \subseteq \liminf A(t_n)$. Now, let $(x(t_{n_k}))_{k \in \mathbb{N}}$ be a subsequence with $x(t_{n_k}) \in A(t_{n_k})$, $k \in \mathbb{N}$ such that $x(t_{n_k}) \rightarrow x(t)$. Passing to the limit for $k \rightarrow \infty$ in

$$\lambda(t_{n_k}) \leq x(t_{n_k})$$

using again the continuity of λ we get $x(t) \in A(t)$, therefore $\limsup A(t_n) \subseteq A(t)$. Analogously $B(t) = \{x(t) \in \mathbb{R} \mid x(t) \leq \mu(t)\}$. In this case, since $\lambda < \mu$ we have $\mathbb{R} = \mathbb{R}_+(A(t) - B(t))$.

The above result is no longer true provided $\lambda \in L^2[0, T]$. Anyway, following [1] if one defines

$$K(\lambda) = \{x \in H^1[0, T] \mid \int_{[0, T]} (x - \lambda)(t) dt \geq 0\}$$

it was proved that $K(\lambda_n) \xrightarrow{M} K(\lambda)$ as $\lambda_n \xrightarrow{w} \lambda$ in $L^2[0, T]$.

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. That result can be viewed as a formalism for the static case. In our future work we shall be concerned on the dynamic case, precisely where the parametric domains and functions are depending on time.

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