GENERATED MOSCO CONVERGENCE FOR SETS

Marcel BOGDAN*

Ph. D. Lecturer, "Petru Maior" University, Târgu-Mureş, Romania

Proceedings of the European Integration between Tradition and Modernity
"Petru Maior" University of Târgu-Mureş
Oktober, 22-23, 2009

Abstract

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. We emphasize some formalisms in normed spaces that generate convergent sequences of sets. A different approach for a kind of restriction sets that proves Mosco convergence is given.

Key words: Parametric problems, Equilibrium problems, Variational inequalities, Mosco convergence, Topological pseudomonotonicity.

There are numerous problems in nonlinear analysis, like scalar and/or vector equilibrium problems, scalar and/or vector variational inequality problems, where parametric domains occur (see [2, 5, 6]).

Let X be a normed space.

Let A, A_n, $n \in \mathbb{N}$ be nonempty sets in X. We shall use the following notations:

$$\liminf A_n = \{x \in X \mid \exists (x_n), x_n \in A_n, \forall n \in \mathbb{N}, x_n \to x\};$$

$$\limsup A_n = \{x \in X \mid \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \to x\};$$

$$w - \limsup A_n = \{x \in X \mid \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \xrightarrow{w} x\},$$

where w denotes the weak convergence in X.

Definition 1.

(PK) *The sequence $(A_n)_{n\in\mathbb{N}}$ is said to converges in the Painlevé-Kuratowski sense to A and notes $A_n \xrightarrow{PK} A$ if*

$$\limsup A_n \subseteq A \subseteq \liminf A_n;$$

*e-mail: bgdmarcel@netscape.net

(M) *The sequence* $(A_n)_{n\in\mathbb{N}}$ *is said to converges in the Mosco sense to* A *and notes* $A_n \xrightarrow{M} A$ *if*

$$w - \limsup A_n \subseteq A \subseteq \liminf A_n.$$

It is clear that if $A_n \xrightarrow{M} A$ then $A_n \xrightarrow{PK} A$.

It is easy to provide examples in $\mathbb{R}$. Let $n \geq 1$, $A_n = \{-1/n, n\}$ for n odd, and $A_n = \{1/n\}$ for n even. Then $A_n \xrightarrow{PK} A$, where $A = \{0\}$.

The parametrization concept can be done for the discrete mode, i.e. the parameter is $n \in \mathbb{N}$ or a continuous mode when the parameter is considered in a topological (or metric) space. It is the last case when the parameter is the time t in an interval $I \subseteq \mathbb{R}$ (see [3]). A slight generalization for sets convergence has been used in this case. Let (X, σ) be a Hausdorff topological spaces, let P (the set of parameters) be also a Hausdorff topological space and let $p_0 \in P$ be fixed. Along with the topology σ, we also consider a stronger topology τ on X. If X is a normed space and $\sigma = \tau =$ norm topology, $D_p \xrightarrow{M} D_{p_0}$ amounts to saying that the sets D_p converge to D_{p_0} in the Painlevé-Kuratowski sense as $p \to p_0$. If X is a normed space and σ is chosen as the weak topology and τ as the norm topology, then obviously, $D_p \xrightarrow{M} D_{p_0}$ implies $D_p \xrightarrow{PK} D_{p_0}$ as $p \to p_0$. It is useful the following form.

Definition 2. *Let* D_p *be subsets of* X *for all* $p \in P$. *The sets* D_p *converge to* D_{p_0} (*and write* $D_p \xrightarrow{M} D_{p_0}$) *as* $p \to p_0$ *if:*

(a) *for every net* $(a_{p_i})_{i\in I}$ *with* $a_{p_i} \in D_{p_i}$, $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *imply* $a \in D_{p_0}$;

(b) *for every* $a \in D_{p_0}$, *there exist* $a_p \in D_p$ *such that* $a_p \xrightarrow{\tau} a$ *as* $p \to p_0$.

In [1] is described a concrete situation in order to generate Mosco convergence. For a reflexive Banach space X, a set-valued mapping $D : X \to 2^X$ is defined by

$$D(x) = D_0 + d(x),$$

where D_0 is a closed convex nonempty subset of X and $d : X \to X$ is a compact map (i.e. a weakly-strongly continuous map).

Proposition 1. ([1], Proposition 1) *For any sequence* $(x_n)_{n\in\mathbb{N}}$, $x_n \xrightarrow{w} x$ *in* X, *then* $D(x_n) \xrightarrow{M} D(x)$.

A partial result from [1], Theorem 1 can be deduced from our result Theorem 1 in [4].

Are there other formalisms that generate Mosco convergence for sets ? Certainly, among them see Lemma 1.4, 1.6 in [6].

This type of convergence was introduced by Umberto Mosco in [6]. His examples were the following:

Example 1. *Let* $X = l^2 = \{y = (y_k)_{k\in\mathbb{N}} \mid \sum_{k=1}^{\infty} |y_k|^2 < +\infty\}$. *Let* B *the unit ball in* X *with respect to the usual norm* $\|y\| = \left(\sum_{k=1}^{\infty} |y_k|^2\right)^{1/2}$. *Denote by* $S_0 = (2B) \cap \{y = (y_k)_{k\in\mathbb{N}} \mid 0 \leq y_k \leq 1, \forall k \in \mathbb{N}\}$ *and* $S_n = (2B) \cap \{y = (y_k)_{k\in\mathbb{N}} \mid 0 \leq y_k \leq 1 + k/n, \forall k \in \mathbb{N}\}$. *One has* $S_n \xrightarrow{M} S_0$.

Let $E = \{e_k \mid k \in \mathbb{N}\}$ *be the canonical basis. Denote by* $C_0 = \operatorname{cl}\operatorname{conv}\{e_1, e_2, ..., e_k, ...\}$ *and* $C_n = \operatorname{cl}\operatorname{conv}\{(1+1/n)e_1, (1+2/n)e_2, ..., (1+k/n)e_k, ...\}$. *One has* $C_n \xrightarrow{M} C_0$.

Now, conform Proposition 1.2.1 in [2], for $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, one has

- $\limsup(A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n$
- $\liminf(A_n \cap B_n) \subseteq \liminf A_n \cap \liminf B_n$.

The question rises: if $A_n \xrightarrow{PK} A$ and $B_n \xrightarrow{PK} B$ when $A_n \cap B_n \xrightarrow{PK} A \cap B$? The same: if $A_n \xrightarrow{M} A$ and $B_n \xrightarrow{M} B$ when $A_n \cap B_n \xrightarrow{M} A \cap B$?

There is an answer given in [7], Proposition 27 along with the constrained condition $X = \mathbb{R}_+(A - B)$.

In [3] the following kind of sets occurs:

$$K(t) = \{x(t) \in \mathbb{R}^m : \lambda(t) \le x(t) \le \mu(t), M(t) \cdot x(t) = b(t)\},$$

where $\lambda, \mu \in C([0,T], \mathbb{R}^m)$ with $\lambda < \mu$ and $b \in C([0,T], \mathbb{R}^l_+)$ are vector-functions and $M \in C([0,T], \mathbb{R}^{l\times m}_+)$ is a matrix-function.

Proposition 2. ([3]) *Let* $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ *be a sequence such that* $t_n \to t$. *Then*

$$K(t_n) \xrightarrow{PK} K(t), \text{ as } n \to \infty.$$

We shall consider a more simple case but a different approach. Define the closed, convex sets

$$A(t) = \{x(t) \in \mathbb{R}^m \mid \lambda(t) \le x(t)\}.$$

Since the product sets enjoys the following properties:

- $\limsup \prod_{k=1}^m A_n^k \subseteq \prod_{k=1}^m \limsup A_n^k$;
- $\liminf \prod_{k=1}^m A_n^k = \prod_{k=1}^m \liminf A_n$,

we can conclude that $A(t_n) \xrightarrow{PK} A(t)$, as $n \to \infty$ if this is proved for $m = 1$.

Let $x(t) \in A(t)$. Define

$$x(t_n) := x(t) - \lambda(t) + \lambda(t_n).$$

We have $x(t_n) \in A(t_n)$ and by the continuity of λ we get $\lim_{n\to\infty} x(t_n) = x(t)$, that is $A(t) \subseteq \liminf A(t_n)$. Now, let $(x(t_{n_k}))_{k\in\mathbb{N}}$ be a subsequence with $x(t_{n_k}) \in A(t_{n_k})$, $k \in \mathbb{N}$ such that $x(t_{n_k}) \to x(t)$. Passing to the limit for $k \to \infty$ in

$$\lambda(t_{n_k}) \le x(t_{n_k})$$

using again the continuity of λ we get $x(t) \in A(t)$, therefore $\limsup A(t_n) \subseteq A(t)$. Analogously $B(t) = \{x(t) \in \mathbb{R} \mid x(t) \le \mu(t)\}$. In this case, since $\lambda < \mu$ we have $\mathbb{R} = \mathbb{R}_+(A(t) - B(t))$.

The above result is no longer true provided $\lambda \in L^2[0,T]$. Anyway, following [1] if one defines

$$K(\lambda) = \{x \in H^1[0,T] \mid \int_{[0,T]} (x-\lambda)(t)\,dt \geq 0\}$$

it was proved that $K(\lambda_n) \xrightarrow{M} K(\lambda)$ as $\lambda_n \xrightarrow{w} \lambda$ in $L^2[0,T]$.

In [4] a strong result has been proved. It consists on the behavior of the solutions of parametric equilibrium problems. That result can be viewed as a formalism for the static case. In our future work we shall be concerned on the dynamic case, precisely where the parametric domains and functions are depending on time.

References

[1] Adly, S., Bergounioux, M., Ait Mansour, M.: Optimal control of a quasi-variational obstacle problem. J. of Global Optim., DOI 10.1007/s10898-008-9366-y.

[2] Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston, Massachusetts (1990).

[3] Barbagallo, A., Cojocaru, M.-G.: Continuity of solutions for parametric variational inequalities in Banach space. J. of Math. Analysis and Appl. 351(2009) 707-720.

[4] Bogdan, M., Kolumbán, J.: Some regularities for parametric equilibrium problems. J. Global Optim. 44(2009) 481-492.

[5] Lucchetti, R., Convexity and Well-Posed Problems, CMS Books in Mathematics. Canadian Mathematical Society, Springer (2006).

[6] Mosco, U., Convergence of convex sets and of solutions of variational inequalities. Adv. in Mathematics 3(1969) 510-585.

[7] Penot, J.-P., Zălinescu, C., Continuity of usual operations and variational convergence. Set-Valued Analysis 11(2003) 225-256.