REPRESENTATIONS OF POSITIVE INTEGERS AS SUM OF SQUARES AND TRIANGULAR NUMBERS

R.Y. Denis [*], [†] S.N. Singh [‡] and S.P. Singh [§¶]

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Abstract

In this paper, using some known summations, we have established the generating functions for the number of representations of a positive integer as the sum of squares of integers and also as the sum of triangular numbers.

Key words and phrases: Sum of squares, sum of triangular numbers, general theta function.

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1 Introduction, notations and definitions

One of the most interesting problems in number theory is the representation of positive integers as sum of squares of integers.Fermat proved that all primes of the form $4n + 1$can be,uniquely, expressed as the sum of two

[*]Department of Mathematics, University of Gorakhpur, Gorakhpur-273009, India
[†]Corresponding author
[‡]Department of Mathematics, University of Gorakhpur, Gorakhpur-273009, India
[§]Department of Mathematics, T.D.P.G. College, Jaunpur-222002, India
[¶]E-mail: ddry@rediffmail.com

squares. Lagrange showed that all positive integers can be represented as sum of four squares and that this number is minimal.
Let $\gamma_k(n)$ stand for the number of ways the positive integer n can be represented as a sum of k squares with representations arising from different signs and from different orders being regarded as distinct.
Since,

$$1 = (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2,$$

we have $\gamma_2(1) = 4$. Also,

$$1 = (\pm 1)^2 + 0^2 + 0^2 = 0^2 + (\pm 1)^2 + 0^2 = 0^2 + 0^2 + (\pm 1)^2$$

so, we have $\gamma_3(1) = 6$.
Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1.$$

The most important special cases of $f(a, b)$ are given by,

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q;-q]_\infty}{[q;-q]_\infty} = \frac{[q^2,-q;q^2]_\infty}{[-q,q;q^2]_\infty},$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty}, \qquad (1)$$

where $[\alpha;q]_\infty = \Pi_{r=0}^{\infty}(1-\alpha q^r)$.
The generating function for $\gamma_k(n)$ is given by

$$\sum_{n=0}^{\infty} \gamma_k(n) q^n = \phi^k(q) = \Big\{ \sum_{n=-\infty}^{\infty} q^{n^2} \Big\}^k.$$

The well known two and four square theorems due to Jacobi are:

$$\gamma_2(n) = 4[d_1(n) - d_3(n)]$$

and

$$\gamma_4(n) = 8 \sum_{d/n;\& \ 4 \ does \ not \ divide \ n} d,$$

where $d_i(n)$ denotes the number of positive divisors of n that are congruent to i (modulo 4).

A triangular number is a number of the form $n(n+1)/2$ for non negative integer n. Let $t_k(n)$ denote the number of representations of n as the sum of k triangular numbers, then generating function for $t_k(n)$ is given by

$$\sum_{n=0}^{\infty} t_k(n)q^n = \psi^k(q) = \left\{\sum_{j=0}^{\infty} q^{j(j+1)/2}\right\}^k,$$

where $\psi(q)$ is given by (1).

The first few triangular numbers are $0, 1, 3, 6, 10, 15, ...$ Thus, $t_2(6) = 3$. Gauss proved that every natural number is the sum of 3 or fewer triangular numbers. Ewell [4] used Jacobi's triple product identity to show that,

$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

Adiga [1] used ${}_1\psi_1$ summation formula due to Ramanujan to show that

$$t_4(n) = \sum_{d/(2n+1)} d.$$

Coper and Lam [3], using the summation formula for ${}_1\psi_1$ derived formulas for $t_2(n), t_4(n), t_6(n)$, and $t_8(n)$. We produce here $t_6(n)$ and $t_8(n)$,

$$t_6(n) = \frac{1}{8} \sum_{d/4n+3} (-1)^{\{(d+1)/2\}} d^2$$

and

$$t_8(n) = \sum_{d/(n+1), d-odd} \left\{\frac{(n+1)}{d}\right\}^3.$$

We define the basic bilateral hypergeometric series as

$${}_r\psi_r \left[\begin{array}{l} a_1, a_2, ..., a_r;\ q;\ z \\ b_1, b_2, ..., b_r \end{array} \right] = \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, ..., a_r]_n \, z^n}{[b_1, b_2, ..., b_r]_n},$$

valid for $\max\{|a_1|, |a_2|, ..., |a_r|, |b_1|, |b_2|, ..., |b_r|, |z|\} < 1$.

2 Main Results

In this section we shall establish certain identities from some known summations and then use these to derive the values of $\gamma_k(n)$ and $t_k(n)$.

(a) Ramanujan's remarkable summation formula for ${}_1\psi_1$ is

$$ {}_1\psi_1\left[\begin{array}{l} a;\ q;\ z \\ b \end{array}\right] = \sum_{n=0}^{\infty} \frac{[a;q]_n z^n}{[b;q]_n} = \frac{[az, q/az, q, b/a; q]_\infty}{[z, b/az, b, q/a; q]_\infty}. \tag{2} $$

Now, replacing q by q^k and then setting $a = q^j, b = q^{k+j}$ and $z = q^i$ in (2), we get

$$ \sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{kn+j}} = \frac{[q^k;q^k]_\infty^2 [q^{i+j}, q^{k-i-j}; q^k]_\infty}{[q^i, q^j, q^{k-i}, q^{k-j}; q^k]_\infty} \tag{3} $$

provided $i, j \not\equiv 0\ mod(k)$.

(i) Now, taking $k = 4$, $i = j = 1$ in (3), we find

$$ \begin{aligned} \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}} &= \sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}} = \frac{[q^2;q^2]_\infty^2}{[q;q^2]_\infty^2} = \psi^2(q) \\ &= \sum_{n=0}^{\infty} t_2(n) q^{2n} \end{aligned} \tag{4} $$

One can notes that (4) implies the following well known identity

$$ t_2(n) = d_1(4n+1) - d_3(4n+1). $$

(ii) Again, taking $k = 4$, $i = 2$ and $j = 1$ in (3), we get

$$ \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}} = \frac{[q^4;q^4]_\infty^2}{[q^2;q^4]_\infty^2} = \psi^2(q^2). \tag{5} $$

Thus we have

$$ \sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{4n+3}} = \psi^2(q^2) = \sum_{n=0}^{\infty} t_2(n) q^{2n}. \tag{6} $$

Now, equating the coefficients of q^{2n} on both sides of (6), we get

$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

Ramanujan's ${}_1\psi_1$ summation can, also, be written as

$$\sum_{n=0}^{\infty} \frac{[q/\beta; q]_n(-\beta q^{-1/2}/z)^n}{[\alpha q; q]_n} + \sum_{n=1}^{\infty} \frac{[1/\alpha; q]_n(-\alpha z q^{1/2})^n}{[\beta; q]_n}$$
$$= \frac{[-zq^{1/2}, -q^{1/2}, q, \alpha\beta; q]_\infty}{[-\alpha z q^{1/2}, -\beta/zq^{1/2}, \alpha q, \beta; q]_\infty}. \tag{7}$$

Differentiating both sides of (7) with respect to z and then setting $z = q^{-1/2}$ Bhargava and Somashekara [2] established the following identity,

$$\sum_{n=0}^{\infty} \frac{(n+1)[q/\alpha; q]_n \alpha^n}{[\beta; q]_{n+1}} + \sum_{n=0}^{\infty} \frac{n[q/\beta; q]_n \beta^n}{[\alpha; q]_{n+1}} = \frac{[q; q]_\infty^3 [\alpha\beta; q]_\infty}{[\alpha, \beta; q]_\infty^2}. \tag{8}$$

Replacing q by q^k and then taking $\alpha = q^i$ and $\beta = q^j$ in (8), we get

$$\sum_{n=0}^{\infty} \frac{(n+1)[q^{k-i}; q^k]_n q^{in}}{[q^j; q^k]_{n+1}} + \sum_{n=0}^{\infty} \frac{n[q^{k-j}; q^k]_n q^{jn}}{[q^i; q^k]_{n+1}}$$
$$= \frac{[q^k; q^k]_\infty^3 [q^{i+j}; q^k]_\infty}{[q^i, q^j; q^k]^2 \infty}; \quad (i, j, i+j < k). \tag{9}$$

Now, taking $k = 2$, $i = j = 1$ in (9), we get

$$\sum_{n=0}^{\infty} \frac{(2n+1)q^n}{1 - q^{2n+1}} = \frac{[q^2; q^2]_\infty^4}{[q; q^2]_\infty^4} = \psi^4(q) = \sum_{n=0}^{\infty} t_{4(n)} q^n$$

which has the arithmetic consequence

$$t_4(n) = \sum_{d/(2n+1)} d,$$

which is a known result due to Legendre.

Ramanujan [6] in Chapter 17, page 139, of the second notebook has mentioned the following identity,

$$\sum_{n=1}^{\infty} \frac{n^3 q^{n-1}}{1-q^{2n}} = \psi^8(q),$$

which can be written as

$$\sum_{n=0}^{\infty} \frac{(n+1)^3 q^n}{1-q^{2n+2}} = \psi^8(q) = \sum_{n=0}^{\infty} t_8(n) q^n$$

which the following arithmetic interpretation,

$$t_{8(n)} = \sum_{d/n+1, d-odd} \left\{\frac{n+1}{d}\right\}^3.$$

(b) Bailey's sum of a well-possed ${}_3\psi_3$ is

$$\sum_{n=-\infty}^{\infty} \frac{[b,c,d;q]_n (q/bcd)^n}{[q/b,q/c,q/d;q]_n} = \frac{[q,q/bc,q/bd,q/cd;q]_\infty}{[q/b,q/c,q/d,q/bcd;q]_\infty}$$

which can be re-written as,

$$1 + \sum_{n=1}^{\infty} \frac{[b,c,d;q]_n (1+q^n)(q/bcd)^n}{[q/b,q/c,q/d;q]_n} = \frac{[q,q/bc,q/bd,q/cd;q]_\infty}{[q/b,q/c,q/d,q/bcd;q]_\infty}. \quad (10)$$

The above relation can also be deduced from the well known basic bilateral analogue of Dixon's summation, (cf. Gasper-Rahman [5]; App.II(II.32)]). Now, taking $b = c = d = -1$ in (10), we get

$$1 + 8\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1+q^n)^2} = \frac{[q;q]_\infty^4}{[-q;q]_\infty^4} = \phi^4(-q). \quad (11)$$

Now, replacing q by $-q$ in (11), we have

$$1 + 8\sum_{n=1}^{\infty} \frac{q^n}{\{1+(-q)^n\}^2} = \phi^4(q). \quad (12)$$

Now, (12) can be interpreted as

$$
\begin{aligned}
\phi^4(q) &= \sum_{n=0}^{\infty}\gamma_4(n)q^n = 1+8\sum_{n=1}^{\infty}\frac{q^n}{\{1+(-q)^n\}^2} = 1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n}\\
&= 1+8\Big\{\sum_{n\geq 1, n-odd}\frac{nq^n}{1-q^n}+\sum_{n\geq 2, n-even}\frac{nq^n}{1+q^n}\Big\}\\
&= 1+8\Big\{\sum_{n\geq 1}\frac{nq^n}{1-q^n}+\sum_{n\geq 2, n-even}\Big(\frac{nq^n}{1+q^n}-\frac{nq^n}{1-q^n}\Big)\Big\}\\
&= 1+8\Big\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n\geq 2, n-even}\frac{2nq^{2n}}{1-q^{2n}}\Big\}\\
&= 1+8\Big\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}\Big\}
\end{aligned}
$$

which implies that

$$
1+\sum_{n=1}^{\infty}\gamma_{4(n)}q^n = 1+8\sum_{4 \ does \ not \ divide \ n}\frac{nq^n}{1-q^n}. \tag{13}
$$

Now, equating the coefficients of q^n on both sides of (13), we get,

$$
\gamma_{4(n)} = 8\sum_{d \ divides \ n \ but \ 4 \ does \ not \ divide \ n} d.
$$

Again, letting $d\to\infty$ in (10), we get

$$
1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n-1)/2}(1+q^n)[b,c;q]_n(q/bc)^n}{[q/b,q/c]_n} = \frac{[q,q/bc;q]_\infty}{[q/b,q/c;q]_\infty}. \tag{14}
$$

Now, setting $b=c=-1$ in (14) and replacing q by $-q$ in it, we get

$$
1+4\sum_{n=1}^{\infty}\frac{(-1)^{n(n-1)/2}q^{n(n+1)/2}}{1+(-q)^n} = \frac{[-q;-q]_\infty^2}{[q;-q]_\infty^2} = \phi^2(q). \tag{15}
$$

Thus, we have

$$\begin{aligned}\phi^2(q) &= \sum_{n=0}^{\infty}\gamma_2(n)q^n = 1+4\sum_{n=1}^{\infty}\left\{\frac{q^{4n-3}}{1-q^{4n-3}}-\frac{q^{4n-1}}{1-q^{4n-1}}\right\}\\ &= \sum_{n=0}^{\infty}\gamma_2(n)q^n = 1+4\sum_{n=0}^{\infty}\left\{\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right\}.\end{aligned}$$

Now, equating the coefficients of q^n on both sides of the above relation, we get Jacobi's two square theorem,

$$\gamma_2(n) = 4\{d_1(n) - d_3(n)\},$$

where $d_i(n)$ is the number of positive divisors of n that are congruent to i (*modulo* 4).

(c) The basic bilateral, (cf.Gasper-Rahman [5]; App.II(II.32)), analogue of Dixon's summation is

$$\sum_{n-\infty}^{\infty}\frac{(1+q^n a^{1/2})[b,c,d;q]_n(qa^{3/2})^n}{(1+a^{1/2})[aq/b,aq/c,aq/d;q]_n(bcd)^n}$$

$$= \frac{[aq,aq/bc,aq/bd,aq/cd,qa^{1/2}/b,qa^{1/2}/c,qa^{1/2}/d,q,q/a;q]_\infty}{[aq/b,aq/c,aq/d,q/b,q/c,q/d,qa^{1/2},q/a^{1/2},qa^{3/2}/bcd;q]_\infty}.$$

Letting $a \to 1$ in (5), we get

$$1+\sum_{n=1}^{\infty}\frac{[b.c.d;q]_n(1+q^n)(q/bcd)^n}{[q/b,q/c,q/d;q]_n} = \frac{[q,q/bc,q/bd,q/cd;q]_\infty}{[q/b,q/c,q/d,q/bcd;q]_\infty}$$

which is (10). Now proceeding on the above lines we can again find the identities (12) and (15) from which $\gamma_4(n)$ and $\gamma_2(n)$ can be formulated. In entry 25 of Chapter 16 of second Notebook, Ramanujan has mentioned the following beautiful identities:

$$\phi(q) + \phi(-q) = 2\phi(q^4)$$

$$\phi(q) - \phi(-q) = 4q\psi(q^8)$$

$$\phi(q)\phi(-q) = \phi^2(-q^2) \tag{16}$$

$$\psi(q)\psi(-q) = \psi(q^2)\phi(-q^2)$$

$$\phi(q)\psi(q^2) = \psi^2(q) \tag{17}$$

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4) \tag{18}$$

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2) \tag{19}$$

$$\phi^4(q) - \phi^4(-q) = 16q\phi^4(q^2). \tag{20}$$

Now, from (20) we have

$$\sum_{k=0}^{\infty} \gamma_4(k)q^k - \sum_{k=0}^{\infty} \gamma_4(k)(-q)^k = 16 \sum_{k=0}^{\infty} \gamma_4(k)q^{2k+1}. \tag{21}$$

Now, equating the coefficients of q^{2k+1} on both sides of (21), we get

$$\gamma_4(2k+1) = 8\gamma_4(k).$$

Further, from (19) we have

$$\sum_{k=0}^{\infty} \gamma_2(k)q^k + \sum_{k=0}^{\infty} \gamma_2(-q)^k = 2 \sum_{k=0}^{\infty} \gamma_2(k)q^{2k}. \tag{22}$$

Now, equating the coefficients of q^{2k} on both sides of (22), we get $\gamma_2(2k) = 2\gamma_2(k)$. Again, (18) leads to

$$\sum_{k=0}^{\infty} \gamma_2(k)q^k - \sum_{k=0}^{\infty} \gamma_2(k)(-q)^k = 8 \sum_{k=0}^{\infty} t_2(k)q^{4k+1}$$

which, on equating the coefficients of q^{4k+1} on both sides, leads to

$$\gamma_2(4k+1) = 4t_2(k).$$

Squaring both sides of (17) we get

$$\phi^2(q)\psi^2(q^2) = \psi^4(q)$$

which leads to

$$\sum_{m=0}^{\infty} \gamma_2(m)q^m \sum_{n=0}^{\infty} t_2(n)q^{2n} = \sum_{k=0}^{\infty} t_4(k)q^k. \tag{23}$$

Now, equating the coefficients of q^{2k} on both sides of (23), we get

$$\sum_{n=0}^{k} \gamma_2(2k-2n)t_2(n) = t_4(2k).$$

Again, squaring both sides of (16), we get

$$\phi^2(q)\phi^2(-q) = \phi^4(-q^2),$$

which can be put in the form

$$\sum_{m=o}^{\infty} \gamma_2(m)q^m \sum_{n=0}^{\infty} \gamma_2(n)(-q)^n = \sum_{k=0}^{\infty} \gamma_4(k)(-1)^k q^{2k}.$$

Now, setting $m = 2k - n$ in the above and equating the coefficients of q^{2k} on both sides,we get

$$\sum_{n=0}^{2k} (-1)^n \gamma_2(n)\gamma_2(2k-n) = (-1)^k \gamma_4(k)$$

which is the same as,

$$\sum_{n=0}^{4k} (-1)^n \gamma_2(n)\gamma_2(4k-n) = \gamma_4(2k).$$

Similarly, scores of other relations can also be established.

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