

On the Postulates for Lattices

Grațiela Laslo

Abstract

This paper is inspired from a series of papers written by J.A. Kalman, in 1955-1959, having the subject the postulates for lattices. It refers especially to systems formed from absorption, idempotence and associativity laws. Following the Kalman's indications from [1], we studied an extended system of axioms, finding all the implications between the subsets of this system.

Introduction

Let us denote by \mathcal{L} the family of all algebraic systems $l = (L, \wedge, \vee)$ consisting of a set L , together with two binary operations on it and let ω be the following set of axioms :

- | | |
|---|---|
| (1) $x \wedge (x \vee y) = x$ | (5) $x \vee (x \wedge y) = x$ |
| (2) $x \vee (y \wedge x) = x$ | (6) $x \wedge (y \vee x) = x$ |
| (3) $(y \vee x) \wedge x = x$ | (7) $(y \wedge x) \vee x = x$ |
| (4) $(x \wedge y) \vee x = x$ | (8) $(x \vee y) \wedge x = x$ |
| (A) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ | (B) $x \vee (y \vee z) = (x \vee y) \vee z$ |
| (C) $x \wedge y = y \wedge x$ | (D) $x \vee y = y \vee x$ |
| (I) $x \wedge x = x$ | (J) $x \vee x = x$ |

and for each $\xi \in \omega$, let \mathcal{L}_ξ be the family of all l in \mathcal{L} such that l obeys all the laws in ξ . Sorokin considered the set ω in [5] §2, and found all the subsets of ω which constitute an independent set of axioms for lattices. For each $\xi \subseteq \omega$, \mathcal{L}_ξ is a family of generalized lattices. Mostly in the years '60 and '70, a few mathematicians studied noncommutative generalizations of lattices: S.I. Matsushita, P. Jordan, M.D. Gerhardt, H. Alfanz and nowadays J. Leech, R.J. Bignall, Gh. Fărcaș, Matthew Spinks, Karin Cvetko-Vah, From these, Gh. Fărcaș and J. Leech have helped permanently author of this paper in studying these structures.

About the postulates for lattices have also been written books we mention here "Axiomele laticilor și algebrilor booleene" ("The Axioms of Lattices and Boolean Algebras") by S. Rudeanu.

In studying noncommutative generalizations for lattices, there were considered different systems of axioms included in. The study of the system ω made by Kalman in [1] is useful even today, as much as the study of an enlarged system denoted by ω_+ , having in addition two axioms (A_0) and (B_0) , which are weaker than the associativity of " \wedge " and " \vee ".

Kalman mentioned that, at the beginning of the study of any family \mathcal{L}_ξ , the following problems arise: P_ξ to find all the subsets of ω that constitute an independent system of axioms for the family \mathcal{L}_ξ and Q_ξ to find all the laws (X) in ω which are obeyed by every l in \mathcal{L}_ξ . Sorkin has solved the problem P_ω and the problem Q_ω is trivial since \mathcal{L}_ω coincide all the class of lattices. Kalman proved results that essentially solve all the problems P_ξ and Q_ξ , with $\xi \subseteq \omega$. He gave a table presenting "what results" from each independent subset of ω . The table has 95 lines and contains 351 such implications.

In [2] and [3] we studied problems connected with the problems P_ξ and Q_ξ with $\xi \subseteq \omega$: the idempotency in systems of the form $\mathcal{L}[- -]$, namely the systems \mathcal{L}_ξ where ξ is constituted from two absorption laws, the problem of commutativity of \wedge and \vee in the systems of the form $\mathcal{L}[- - -]$, the relations among the systems of the form $\mathcal{L}[- - -]$. This study was done using direct proofs and counterexamples.

Also Kalman considered weaker axioms than associativity :

$$(A_0) x \wedge (y \wedge x) = (x \wedge y) \wedge x \quad \text{and} \quad (B_0) x \vee (y \vee x) = (x \vee y) \vee x$$

If we add to the system ω these two axioms, then the study of $\omega_+ = \omega \cup \{(A_0), (B_0)\}$ is again interesting. For instance we are interested to find out what absorption, idempotency, commutativity axioms result from a certain system of absorption, idempotency, commutativity axioms. We want to see when the associativity axioms are essential in obtaining a certain result and when can they be replaced by weaker axioms. The study of this extended system of axioms was proposed by Kalman in [1] §3 and is made by me in the present paper.

In the first section we make some remarks on the results obtained in [2] and [3] and the results obtained by Kalman concerning the subsets of absorption identities. The second section (Closure operations) presents some elementary results concerning closure operations on a given complete lattice. In this second section we define also the closure operator a , which will be essential for solving the proposed problem.

The third section presents the compatibility of a with a group of automorphisms. The fourth section contains the main results of the paper. The

preparing results which are the four lemmas were given by Kalman in [1] and proved here by me. These helped me to establish Theorem 1 and Theorem 2 for ω_+ and make the program for proving them.

1 The seven classes of noncommutative lattices

As we said in the previous section, in [2] were presented some classes of algebraic structures generalizing lattices, of the form $\mathcal{L}[- -]$, namely \mathcal{L}_ξ where ξ is constituted from four absorption laws. More precisely were considered groups of four absorptions, two having the operation \wedge outside the brackets and the other two, the operation \vee outside the brackets. For instance

$$a \wedge (a \vee b) = a, \quad (a \vee b) \wedge a = a, \quad a \vee (a \wedge b) = a, \quad (b \wedge a) \vee a = a.$$

We attempted to answer how many groups of such absorption laws we have, namely how many essentially different classes of noncommutative generalizations of lattices define they. We considered the following relation between two systems of the described type:

$$S_1 \sim S_2 \Leftrightarrow S_1 = S_2 \quad \text{or} \quad (S_1)^\wedge = S_2 \quad \text{or} \quad (S_1)^\vee = S_2 \quad \text{or} \quad (S_1)^{\wedge\vee} = S_2.$$

that is: S_1 is equivalent with S_2 iff S_1 is equal to S_2 or S_2 can be obtained from S_1 by interchanging the performing order of " \wedge ", or " \vee ", or of both operations. I found twelve equivalence classes having the representatives: $S'_\wedge, S'_\vee, S'_{\wedge\vee}, S'_{\vee\wedge}, S'_{\wedge\wedge}, S'_{\vee\vee}, S'_{\wedge\vee\wedge}, S'_{\vee\wedge\vee}, S'_\wedge, S'_{\wedge\vee}, S'_\vee, S'_{\vee\wedge}$ (see tabel 1).

From these $S'_{\wedge\vee\wedge}$, is the strongest because the algebraic structures defined by $S'_{\wedge\vee\wedge}$ has both operations commutative, S'_\wedge and S'_\vee have just one operation commutative. From these twelve, the last four are the dual of other four: $S'_\wedge = (S'_\vee)^*$, $S'_{\wedge\vee} = (S'_{\vee\wedge})^*$, $S'_{\wedge\wedge} = (S'_{\vee\vee})^*$, $S'_{\vee\vee} = (S'_\wedge)^*$. (* means the dual of). Thus we have in fact seven essentially different classes of noncommutative lattices.

Kalman in his paper [1] considered different the equivalence relation between the systems of axioms:

$$S_1 \sim S_2 \Leftrightarrow \exists \sigma \in P \text{ such that } S_2 \text{ is the image of } S_1 \text{ by } \sigma.$$

Here P is a group of permutations generated by two certain permutation of the axioms from ω . In fact, the Kalman's definition can be described thus:

$$S_1 \sim S_2 \Leftrightarrow S_1 = S_2 \quad \text{or} \quad (S_1)^\wedge = S_2 \quad \text{or} \quad (S_2)^\wedge \quad \text{or} \quad (S_1)^{\wedge\vee} = S_2 \quad \text{or} \\ S_1^* = S_2 \quad \text{or} \quad (S_1^*)^\wedge = S_2 \quad \text{or} \quad (S_2^*)^\wedge \quad \text{or} \quad (S_1^*)^{\wedge\vee} = S_2$$



For instance, if we want to see the equivalence class of the system formed from the axioms (1) and (2), system denoted by [12], we know that the axiom (1) can become any other absorption axiom, and also (2) by the following sketch:

Thus, the equivalence class of [12] is $[12] = \{[12], [85], [34], [67], [56], [41], [78], [23]\}$.

If we examine the Kalman's table 2 from [1], he obtained 12 equivalence classes having representatives with four absorption laws. We present below an extract from this table, containing them:

| | | | |
|-----|------|----------------------------|--------------------------------|
| 1. | 1234 | $\overline{S'_{\wedge v}}$ | |
| 2. | 1235 | $\overline{S_{\wedge v}}$ | and $\overline{S_{\wedge v}}$ |
| 3. | 1236 | of type | $3 + 1$ |
| 4. | 1237 | $\overline{S_{\wedge v}}$ | and $\overline{S_{\wedge v}}$ |
| 5. | 1256 | $\overline{S_{\wedge v}}$ | |
| 6. | 1257 | of type | $3 + 1$ |
| 7. | 1258 | $\overline{S_{\wedge v}}$ | and $\overline{S_{\wedge v}}$ |
| 8. | 1267 | $\overline{S'_{\wedge v}}$ | and $\overline{S'_{\wedge v}}$ |
| 9. | 1268 | of type | $3 + 1$ |
| 10. | 1278 | $\overline{S_{\wedge v}}$ | |
| 11. | 1357 | $\overline{S_{\wedge v}}$ | |
| 12. | 1368 | of type | $3 + 1$ |

Table 1

From these twelve, four are of type 3+1 (having three absorption with the same operation outside the brackets). The other eight are representatives for the equivalence classes we found, in[2], as it is indicated in table 1. As it is mentioned in all the papers [1], [2], [3] there are examples that prove that these classes of noncommutative generalizations are distinct.

2 Closure operations

Let us consider an operator $a : \mathcal{P}(\omega_+) \rightarrow \mathcal{P}(\omega_+)$ defined by:

$$a\xi = \{(X) \in \omega_+ \mid (X) \text{ is obeyed in any } l \in \mathcal{L}_\xi\}.$$

We can easily verify that it is a closure operator on $\omega_+ = \omega \cup \{A_0, B_0\}$ (defined in the Introduction). In order to establish the exact value of $a\xi$, for every $\xi \subseteq \omega_+$, we will need other closure operators, which "approximate" up and down our operator a .

Our discussion from this section takes place in the general frame of a complete lattice with greatest element V , and it will be applied in the next sections for the complete lattice of systems of axioms $\mathcal{P}(\omega_+)$.

Following the Kalman ideas, we will consider a complete lattice L with greatest element V , G a group of lattice automorphisms $g : L \rightarrow L$ and \mathcal{C} the set of all closure operations $c : L \rightarrow L$, which are "compatible with G ", i.e. which are such that $xgc = xcg$, for all x and g in G . If we define an order relation " \leq " by: $c \leq c'$ if and only if $xc \leq xc'$ for all x in L , it is easy to verify that \mathcal{C} becomes a complete lattice. Let \mathcal{Z} be the set of all subsets Z of L which are such that (i) $V \in Z$ (ii) if $x \subseteq Z$, then $\text{Inf } X \in Z$ and (iii) if $x \in Z$, then $xg \in Z$ for all g in G . The set \mathcal{Z} becomes a complete lattice when for Z, Z' in \mathcal{Z} we set $Z \subseteq Z'$ if and only if $Z \subseteq Z'$ (set theoretic inclusion). Also a dual isomorphism ψ of G onto \mathcal{Z} may be defined by setting: $c\psi = \{x \mid x \in L \text{ and } x = xc\}$.

The inverse dual isomorphism ψ is given by:

$$x(Z\psi^{-1}) = \text{Inf}\{y \mid y \in Z \text{ and } y \geq x\}$$

If c_0 is a partially defined unary operation on L i.e. a mapping of some subset L_0 of L into L , and if c in \mathcal{C} is given by:

$$c = \text{Inf}\{b \mid b \in \mathcal{C} \text{ and } xbc_0 \text{ for all } x \in L_0\},$$

it can be easily verified that $c : L \rightarrow L$ is a closure operator. We will call c_0 a "G-support" of c . If Z_0 is any subset of L , and if Z in \mathcal{Z} is given by:

$$Z = \text{Inf}\{W \mid W \in \mathcal{Z} \text{ and } W \supseteq Z_0\},$$

we will call Z_0 a "G base" of the closure operation $Z\psi^{-1}$. If c is any closure operation on L and $x \in L$, we will say that x is " c "-closed if $x = xc$ and that x is " c -independent" if no y in L is such that $y < x$ and $yc = xc$.

Remark 1. If $c_1, c_2 : L \rightarrow L$ are two closure operations such that $c_1 \leq c_2$, then, for any $\xi \in L$,

- a) ξ is c_1 -dependent $\Rightarrow \xi$ is c_2 -dependent
- b) ξ is c_2 -dependent $\Rightarrow \xi$ is c_1 -independent.

Indeed, suppose there exist $y < \xi$ such that $yc_1 = \xi c_1$. Then we have $y < \xi \leq \xi c_1 = yc_1 \leq yc_2$, thus $\xi \leq yc_2$. Applying c_2 we have $\xi c_2 \leq yc_2$. The converse, inequality is obvious since $y < \xi$. Thus $\xi c_2 = yc_2$ and ξ is c_2 dependent

b) Results from a).

3 The compatibility of a with a group of lattice - automorphisms

On the family \mathcal{L} of all algebraic systems $l = (L, \wedge, \vee)$ consisting of all the set L together with binary operations \wedge and \vee , Kalman considered the transformations Π and ρ :

$$\pi : \mathcal{L} \rightarrow \mathcal{L}, \rho : \mathcal{L} \rightarrow \mathcal{L}, \forall l = (L, \wedge, \vee) \in \mathcal{L}, l\Pi = (L, \wedge_\Pi, \vee_\Pi), l\rho = (L, \wedge_\rho, \vee_\rho)$$

where,

$$(9) x \wedge_\pi y = y \vee x, x \vee_\pi y = x \wedge y, \forall x, y \in L$$

$$(10) x \wedge_\rho y = x \vee y, x \vee_\rho y = x \wedge y, \forall x, y \in L.$$

It is easy to verify that $\rho\pi = \pi^3\rho$ and $\Pi^4 = \rho^2 = \varepsilon$ (the identity transformation). The transformation Π and ρ generate in the subgroup of all transformations on L , a subgroup Γ and all the elements of Γ can be written in at least one way in the form $\Pi^m\rho^n$, $m \in \{0, 1, 2, 3\}$, $n \in \{0, 1, 2\}$.

Kalman also considered two permutations p and q in the permutation group of the elements $1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, I, J, A_0, B_0$. We will consider in the same way two permutation p and q of the elements $1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, I, J, A_0, B_0$:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 2 & 3 & 4 & 1 & 8 & 5 & 6 & 7 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$

By the fact p correspond to the permutation Π we meant that p indicates the correspondence between the axioms fulfilled in an algebraic structure $l \in \mathcal{L}$ and the correspondent axioms that hold in $l\Pi$.

Analogously we determine q . It is easy to verify that, for all $l \in \mathcal{L}$ and $(X) \in \omega_+$; l obeys $(X) \Leftrightarrow l\Pi$ obeys $(X)p \Leftrightarrow l\rho$ obeys $(X)q$. Let P be the subgroups generated by p and q in the group of all permutations of the elements of ω_+ . It is easily seen that the elements of p and q verify: $p^4 = q^2 = e$ and $qp = p^3q$.

If follows that the elements of p are precisely of the form $p^m q^n$, $m \in \{0, 1, 2, 3\}$, $n \in \{0, 1, 2\}$ and that the mapping $\lambda : P \rightarrow \Gamma$, $\lambda(p^m q^n) = \Pi^m \rho^n$, $m \in \{0, 1, 2, 3\}$, $n \in \{0, 1, 2\}$ is an homomorphism of P onto Γ (we will see in §4 that λ is in fact an isomorphism).

We give below the permutation subgroup generated by p and q

| | | | | | | | | | | | | | | | | |
|--------|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|-----|-------|-------|
| r | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | A | B | C | D | I | J | A_0 | B_0 |
| e | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | A | B | C | D | I | J | A_0 | B_0 |
| p | 2 | 3 | 4 | 1 | 8 | 5 | 6 | 7 | B | A | D | C | J | I | B_0 | A_0 |
| p^2 | 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 | A | B | C | D | I | J | A_0 | B_0 |
| p^3 | 4 | 1 | 2 | 3 | 6 | 7 | 8 | 5 | B | A | D | C | J | I | B_0 | A_0 |
| q | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 | B | A | D | C | J | I | B_0 | A_0 |
| pq | 6 | 7 | 8 | 5 | 4 | 1 | 2 | 3 | A | B | C | D | I | J | A_0 | B_0 |
| p^2q | 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 | B | A | D | C | J | I | B_0 | A_0 |
| p^3q | 8 | 5 | 6 | 7 | 2 | 3 | 4 | 1 | A | B | C | D | I | J | A_0 | B_0 |

Table 2.

Using the following:

- an axiom (X) is true in $l \Leftrightarrow$ the axiom $(X)p$ is true in $l\Pi = l(p\lambda)$
- an axiom (X) is true in $l \Leftrightarrow$ the axiom $(X)q$ is true in $l\rho = l(q\lambda)$
- λ is a homomorphism,

the following lemma hold:

Lemma 1. For all $l \in \mathcal{L}$, (X) in ω_+ and $r \in P$, l obeys the law (X) if and only if $l(r\lambda)$ obeys the law $(X)r$.

If we choose a permutation $r \in P$, to each subset of axioms from ω_+ we can associate the corresponding subset of axioms, by r . Thus we have defined a transformation μ :

$$\xi(r\mu) = \{(Y) \mid \exists r \in P \text{ and } \exists (X) \in \xi \text{ and such that } (Y) = (X)r\}$$

having the domain P . $r\mu$ is a lattice automorphism of the Boolean algebra $P(\omega_+)$. If we denote by G the image of μ , we have that μ is an isomorphism of P onto the group of automorphisms of $P(\omega_+)$.

Let's choose again a permutation $r \in P$. We notice that the operator a defined in §2 is, by it's definition, compatible with any transformation which associates to a $\xi \subseteq \omega_+$ the resulting subset $(\xi)r$ (image of the set ξ by

r), namely $[(\xi)r]a = (\xi a)r$. On the other side, from definition of μ we have $\xi(r\mu) = (\xi)r$. Thus,

$$[\xi(r\mu)]a = [(\xi)r]a = (\xi a)r = (\xi a)(r\mu), \forall r\mu \in G,$$

namely we have:

Lemma 2. The closure operation a is compatible with G .

In the final of this paragraph we will define on ω_+ the relation: if $\xi, \eta \subseteq \omega_+$ are such that $\eta = \xi(r\mu)$ for some $r \in P$. we will call ξ and η "congruent" subsets of ω_+ and it follows from Lemma 2 that, if a subset ξ of ω_+ is a -closed, [a -independent] then every η congruent to ξ is a -closed [a -independent].

4 Main result

If a partially defined unary operation c_0 on a given set has domain $\{\xi_1, \xi_2, \dots, \xi_n\}$ and we have $\xi_i c_0 = \eta_i, i = \overline{1, n}$, we will say that c_0 has "defining relations" $\xi_1 \rightarrow \eta_1, \xi_2 \rightarrow \eta_2, \dots, \xi_n \rightarrow \eta_n$. If the distinct elements of a nonempty subset ξ of ω are $(X_1), \dots, (X_n)$, we will write $\xi = [X_1 X_2 \dots X_n]$. Let a_0 be the partially defined operation on the subset of ω which has defined relations

$$[A] \rightarrow [A_0], [C] \rightarrow [A_0], [12] \rightarrow [J], [15] \rightarrow [J], [1C] \rightarrow [8], [1D] \rightarrow [6], \\ [17] \rightarrow [I], [123] \rightarrow [8], [127A_0] \rightarrow [8], [1267B_0] \rightarrow [D] \quad \text{and} \quad [1368BJ] \rightarrow [D],$$

and let a_1 be the closure operation on the subsets of ω which has G -support a_0 . The following lemma hold:

Lemma 3. $a_1 \leq a$.

Proof. It is sufficient to prove $\xi a \supseteq \xi a_0$ for each ξ in the domain of a_0 .

In the presence of associativity (A) or commutativity (C), the axiom A_0 : $(x \wedge y) \wedge x = x \wedge (y \wedge x)$ is obviously fulfilled. The following six implications and the last one are true by Lemma 3 from [1]. We must prove $[127A_0] \rightarrow [8]$ and $[1267B_0] \rightarrow [D]$.

First we must prove using (1), (2), (7) and (A_0) that (8): $(x \vee y) \wedge x = x$ is fulfilled. From [12] results $[J]$ and from $[1J]$ results (I). In any $l = (L, \wedge, \vee)$, from $\mathcal{L}[127A_0]$, for any $x, y \in L$, using A_0 we have first:

$$[x \wedge (x \vee y)] \wedge x = x \wedge [(x \vee y) \wedge x]$$

The member from left is equal to x by 1. After that we apply $\vee[(x \vee y) \wedge x]$ and thus:

$$x \vee [(x \vee y) \wedge x] = (x \vee y) \wedge x.$$

But by (2) the member from left is equal to x , and thus we obtain that (8) is true in L .

We will prove $[1267B_0]$ implies $[D]$.

$$x \vee y \stackrel{7}{=} [y \wedge (x \vee y)] \vee (x \vee y) \stackrel{6}{=} y \vee (x \vee y) \stackrel{B_0}{=} (y \vee x) \vee y.$$

Analogously $y \vee x = (x \vee y) \vee x$.

Using these, we have:

$$x \vee y = (y \vee x) \vee y = (y \vee x) \vee [y \wedge (y \vee x)] \stackrel{2}{=} y \vee x.$$

□

Remark 2. The defining relations $[1267B] \rightarrow [D]$ and $[127A] \rightarrow [8]$ from the study of ω , in [1], were replaced, after Kalman's idea with $[1267B_0] \rightarrow [D]$ and $[127A_0] \rightarrow [B]$. Thus the associativity appears just in two of the defining relations of a_0 .

Remark 3. We remark that the value of the operator a_1 can be calculated. We consider the defining relations of a_0 and their permutation obtained by table 2. There are 66 distinct relations, the set of which will be denoted by S . We consider then the operator $c : P(\omega_+) \rightarrow P(\omega_+)$ which acts as follows on a given $\varepsilon \subseteq \omega_+$: adds the conclusion of each from the 66 relations, if the respective hypothesis is found in ξ , replacing after that each time ξ with the result system. It's obvious that, there exists a natural number which depends of the given ξ , $n(\xi)$ such that $c^{n(\xi)}(\xi) = c^{n(\xi)+1}(\xi)$. If we consider $n = \sup_{\xi \in P(\omega_+)} n(\xi)$, then , for any $\xi \subseteq \omega_+$

$$(11) \quad \xi a_1 = \xi c^n$$

since c^n is a closure operator, verifies $c^n \geq a_0$ on the domain of a_0 and it is the least with these properties.

We will consider now a few algebraic structures (L, \wedge, \vee) which will be counterexamples for certain implications between the subsets of ω_+ . Kalman indicated in [1] the following examples (the sequences mean the rows of the

corresponding multiplication tables for \wedge and \vee):

$$\begin{aligned}
 L_1 &= \{0, 1, 2, 3, 4\}, 00000 01111 01202 01033 01234 01234 11234 22244 33434 44444 \\
 L_2 &= \{0, 1, 2, 3, 4\}, 00000 01111 01202 01133 01234 01234 11234 22244 33434 44444 \\
 L_3 &= \{0, 1, 2\}, 000 011 012 022 112 222 \\
 L_4 &= \{0, 1\}, 01 01 00 11 \\
 L_5 &= \{0, 1, 2\}, 000 012 012 022 111 222 \\
 L_6 &= \{0, 1, 2\}, 010 011 012 022 212 222 \\
 L_7 &= \{0, 1\}, 01 01 00 11 \\
 L_8 &= \{0, 1\}, 01 10 00 00 \\
 L_9 &= \{0, 1\}, 01 11 00 00 \\
 L_{10} &= \{0, 1, 2\}, 000 011 012 022 212 222 \\
 L_{11} &= \{0, 1, 2\}, 000 011 012 012 222 222 \\
 L_{12} &= \{0, 1, 2\}, 000 001 002 012 222 222 \\
 L_{13} &= \{0, 1, 2, 3\}, 0000 0111 0122 0123 0233 3133 3323 3333
 \end{aligned}$$

Remark 4. a) L_2 doesn't verify A_0 . Indeed for $x = 2$ and $y = 3$, $(x \wedge y) \wedge x = x \wedge (y \wedge x) \Leftrightarrow 0 = 1$.

b) L_3 doesn't verify B_0 . Indeed, for $x = 1$, $y = 0$, we have $(x \vee y) \vee x = x \vee (y \vee x) \Leftrightarrow 1 = 2$.

c) L_{14} satisfies $[1368ACIJA_0B_0]$ and doesn't satisfies the rest of the axioms ω_+ . Indeed (L_{14}, \wedge) is the restrictive semigroup of the chain $0 \leq 1 \leq 2 \leq 3$ and it verifies $[ACIA_0]$. The rest it is easy to verify.

Let z_0 be the following family of subsets of ω_+ .

$$\begin{aligned}
 \aleph_1 &= [12345678BCDIJA_0B_0] \\
 \aleph_2 &= [12345678BDIJB_0] \\
 \aleph_3 &= [123568ACIJA_0] \\
 \aleph_4 &= [123578ABIJA_0B_0] \\
 \aleph_5 &= [12358ABIJA_0B_0] \\
 \aleph_6 &= [123678ABDIJA_0B_0] \\
 \aleph_7 &= [1258ABCIIJA_0B_0]
 \end{aligned}$$

$$\begin{aligned}
\aleph_8 &= [1368ABCD A_0 B_0] \\
\aleph_9 &= [1368ABCD I A_0 B_0] \\
\aleph_{10} &= [1368ABCD I J A_0 B_0] \\
\aleph_{11} &= [1368ABC I A_0 B_0] \\
\aleph_{12} &= [17ABA_0 B_0] \\
\aleph_{13} &= [1368AC I J A_0 B_0]
\end{aligned}$$

and let a_2 be the closure operation on the subsets of which has G -base Z_0 .

Lemma 4. $a \leq a_2$.

Proof. Since Z_0 is a G base of a_2 , we have:

$$a_2\psi = Z = \text{Inf}\{W \mid W \in 3\mathcal{Z} \text{ and } W \supseteq Z_0\}$$

and in the same time we know from the definition of ψ that Z is the set of all elements from $P(\omega_+)$ which are a_2 -closed, thus, for any $\xi \subseteq \omega_+$,

$$(12) \xi a_2 = \cap\{y \in Z \mid y \supseteq \xi\}$$

We have to prove $\xi a \leq \cap\{y \in Z \mid y \supseteq \xi\}$, $\forall \xi \subseteq \omega_+$.

We notice from (12) and from definition of \mathcal{Z} that

$$Z = Z_0 \cup \bigcup_{g \in G} g(Z_0) \cap \{\omega_+\} \cup F,$$

where F denotes the sets of the form $\text{inf } X$, where $X \subseteq Z_0 \cup \bigcup_{g \in G} g(Z_0) \cup \{\omega_+\}$.

Thus

$$(13) \xi a_2 = \cap\{y \in Z_0 \cup \bigcup_{g \in G} g(Z_0) \cup \{\omega_+\} \mid y \supseteq \xi\}$$

If we prove that all the elements considered in the last expression are a -closed, then the lemma is true, since each such y will satisfy

$$y = ya \supseteq \xi a.$$

The element ω_+ is obviously a -closed, and if the elements of Z_0 are a -closed, then also the elements of $g(Z_0)$ are a -closed, since a is compatible with any $g \in G$. The fact that the elements of Z_0 are a -closed results from the fact that the algebraic structures L_i , $i = \overline{1, 13}$ presented in this paragraph verify $[\aleph_i]$ but don't verify $\omega_+ \setminus \aleph_i$. \square

Remark 5. We remark that the values of the operator a_2 can be calculated. If we consider the systems $\aleph_1, \aleph_2, \dots, \aleph_{13}$ and the permuted systems obtained from them by table 2, we obtain 104 systems $\aleph_1, \aleph_2, \dots, \aleph_{104}$. By definition of G (see §3) we have then:

$$(14) \xi a_2 \cap \{y \in \{\aleph_1, \aleph_2, \dots, \aleph_{104}\} \mid y \supseteq \xi\}$$

We now state the theorems which are the main results of this paper.

Theorem 1. The operation a_0 is a G -support of a and the family Z_0 is a G base of a .

Proof. To prove theorem 1 it will be sufficient to show that $a_1 = a_2$. Then, by Lemma 3 and 4, we will have $a_1 = a = a_2$, completing the proof.

Let us consider $\theta \subseteq \omega_+$, arbitrary systems of axioms.

The computer programm calculates θa_1 and θa_2 by (11) and (13) and compares the results. For any $\theta \subseteq \omega_+$ we have $\theta a_1 = \theta a_2$ and thus $a_1 = a = a_2$. \square

Let's denote by $\bar{\theta}$ the common value of θa_1 and θa_2 .

Theorem 2. A subset of ω_+ is a -independent if and only if it is congruent to one of the subsets θ listed in column θ of table 3. The entry in the row of a certain θ and column $\bar{\theta}$ of table 3 is θa .

Note It may easily be checked that each entry in column θ of table 3 is the lexicographically first element of it's congruence class. Thus, no two subsets in column 0 are congruent to each other.

Proof. Since for each $\theta \subseteq \omega_+$ we know the exact value of θa as we explained in the proof of theorem 1 means that we can establish, by a procedure with an element $\theta \subseteq \omega_+$ is a -independent. Calculating the value of a for the subsystems of θ . \square

5 About the programm

The programm has got a few functions. The most important are:

- a function "minim" which receive a sequence representing a system of axioms, calculates the elements from the same congruence class using table 2, and returns the least sequence in lexicographical order.
- a function "aplică t" which calculates a_1 for a given sequence θ , using (11).
- a function "aplică 2" which calculates a_2 for a given sequence θ , using (13). The matrix \aleph having the rows $\aleph_1, \dots, \aleph_{104}$ is taken from the main programm and it is generated using table 2 by another function
- a function "verifindep" which verifies if a given system is a -independent.

The main program generates the subsets of ω_+ in lexicographical order. When each system θ is formed, it is "minimized", by the function "minim". After that the program verifies the independency of the resulted system, called "min." We can renounce first at verifying the independency and we want to see first that $a_1 = a_2$ as we explained in the proof of the Theorem 1. In this way we follow the logic order of the ideas. In the case when we verify the independency,

when the system θ is independent it is put in a list, introducing it where it is its place in lexicographical order. After the last element $\xi \subset \omega_+$ has been generated (this is $[B_0]$) and it is verified its independency, the program starts to print the list of a -independent systems. After printing $\theta \subseteq \omega_+$ from the list, it calculates θa_1 by the function "aplicat" and θ_2 by the function "aplica 2", verifies if θa_1 and θa_2 coincide and if not, it gives us a message and stops running. No such message has been received and thus $\theta a_1 = \theta a_2$.

In the case of the coincidence, it prints θa_1 and goes further to the next θ from the list.

The table 3, of the results, contains 599 rows for all the 599 independent systems found in ω_+ .

For a simple writing of the results, the program prints the letters "k" and "l" instead of notations " (A_0) " and " (B_0) ". The axioms (A) , (B) , (C) , (D) , (I) , (J) are denoted in the list of the results with small letters.

Let's interpret the results for two systems of axioms.

The system [1234]:

-is independent since appears in the first column of the tabel 3.

- implies the axioms 1, 2, 3, 4, 5, 6, 7, 8, I , J and no other axioms from ω_+ .

- together with the axiom (A) , implies [12345678ACIJA₀].

-together with the axiom (C) , implies [12345678CIJA₀].

-together with the axiom (A_0) (written in the tabel as k), implies [12345678CIJA₀]

namely the same system of axioms as in the case if we had added (C) , and the same system of absorption, commutativity, and idempotence axioms as in the case if we had added (A) .

The system [1368]:

-is independent since appears in the first column of the tabel 3.

-implies the axioms 1, 3, 6, 8 and no other axioms from ω_+ .

- together with the axiom (A) it implies [1368AA₀].

- together with the axiom (C) it doesn't form an independent subsystem and we must find a subsystem of [1368C] which is independent and implies these axioms. We find [13C], which implies [1368CA₀] and no other axioms beside these.

-together with the axiom (A_0) , it forms an independent system, and it implies just the same system [1368A₀]. Thus, from the point of view of

absorption, commutativity and idempotence axioms that result, we have the same result if we add (A) or (A_0) , but we don't have the same result if we add (C) or (A_0) .

| tetha | tetha-bar |
|---------|------------------|
| 1 | 1 |
| 12 | 12ij |
| 123 | 1238ij |
| 1234 | 12345678ij |
| 1234a | 12345678acijk |
| 1234ab | 12345678abcdijkl |
| 1234al | 12345678acdijkl |
| 1234c | 12345678cijk |
| 1234k | 12345678cijk |
| 1234kl | 12345678cdijkl |
| 1235 | 12358ij |
| 12356 | 123568ij |
| 123567 | 12345678ij |
| 12356a | 123568acijk |
| 12356ab | 12345678abcdijkl |
| 12356al | 12345678acdijkl |
| 12356b | 12345678bdijl |
| 12356bk | 12345678bcdijkl |
| 12356k | 123568cijk |
| 12356kl | 12345678cdijkl |
| 12356l | 12345678dijl |
| 12357 | 123578ij |
| 12357a | 123578aijk |
| 12357ab | 123578abijkl |
| 12357al | 123578aijkl |
| 12357b | 123578bijl |
| 12357bk | 123578bijkl |
| 12357k | 123578ijk |
| 12357kl | 123578ijkl |
| 12357l | 123578ijl |
| 1235a | 12358aijk |
| 1235ab | 12358abijkl |
| 1235al | 12358aijkl |
| 1235b | 12358bijl |
| 1235bd | 12345678bdijl |
| 1235bk | 12358bijkl |
| 1235d | 12345678dijl |
| 1235k | 12358ijk |
| 1235kl | 12358ijkl |
| 1235l | 12358ijl |
| 1236 | 12368ij |
| 12367 | 123678ij |
| 1236a | 12368aijk |
| 1236ab | 123678abdijkl |

| | |
|--------|------------------|
| 1236a1 | 123678adijkl |
| 1236b | 123678bdijl |
| 1236bk | 123678bdijkl |
| 1236k | 12368ijk |
| 1236kl | 123678dijkl |
| 1236l | 123678dijl |
| 1237 | 12378ij |
| 1237a | 12378aijk |
| 1237ab | 12378abijkl |
| 1237ac | 12345678acijk |
| 1237al | 12378aijkl |
| 1237b | 12378bijl |
| 1237bk | 12378bijkl |
| 1237c | 12345678cijk |
| 1237k | 12378ijk |
| 1237kl | 12378ijkl |
| 1237l | 12378ijl |
| 123a | 1238aijk |
| 123ab | 1238abijkl |
| 123abc | 12345678abcdijkl |
| 123ac | 123568acijk |
| 123acl | 12345678acdijkl |
| 123al | 1238aijkl |
| 123b | 1238bijl |
| 123bc | 12345678bcdijkl |
| 123bd | 123678bdijl |
| 123bk | 1238bijkl |
| 123c | 123568cijk |
| 123cl | 12345678cdijkl |
| 123d | 123678dijl |
| 123k | 1238ijk |
| tetha | tetha-bar |
| 123kl | 1238ijkl |
| 123l | 1238ijl |
| 125 | 125ij |
| 1256 | 1256ij |
| 1256a | 1256aijk |
| 1256ab | 1256abijkl |
| 1256al | 1256aijkl |
| 1256k | 1256ijk |
| 1256kl | 1256ijkl |
| 1257 | 1257ij |
| 1257a | 12578aijk |
| 1257ab | 12578abijkl |
| 1257al | 12578aijkl |
| 1257b | 1257bijl |
| 1257bk | 12578bijkl |
| 1257k | 12578ijk |
| 1257kl | 12578ijkl |

| | |
|--------|------------------|
| 1257l | 1257ijl |
| 1258 | 1258ij |
| 1258a | 1258aijk |
| 1258ab | 1258abijkl |
| 1258al | 1258aijkl |
| 1258b | 1258bijl |
| 1258bd | 12345678bdijl |
| 1258bk | 1258bijkl |
| 1258d | 12345678dijl |
| 1258k | 1258ijk |
| 1258kl | 1258ijkl |
| 1258l | 1258ijl |
| 125a | 125aijk |
| 125ab | 125abijkl |
| 125abd | 12345678abcdijkl |
| 125ad | 12345678acdijkl |
| 125al | 125aijkl |
| 125b | 125bijl |
| 125bd | 124567bdijl |
| 125bdk | 12345678bcdijkl |
| 125bk | 125bijkl |
| 125d | 124567dijl |
| 125dk | 12345678cdijkl |
| 125k | 125ijk |
| 125kl | 125ijkl |
| 125l | 125ijl |
| 126 | 126ij |
| 1267 | 1267ij |
| 1267a | 123678aijk |
| 1267ab | 123678abdijkl |
| 1267ac | 12345678acijk |
| 1267al | 123678adijkl |
| 1267b | 1267bdijl |
| 1267bk | 123678bdijkl |
| 1267c | 12345678cijk |
| 1267k | 123678ijk |
| 1267kl | 123678dijkl |
| 1267l | 1267dijl |
| 1268 | 1268ij |
| 1268a | 1268aijk |
| 1268ab | 1268abijkl |
| 1268al | 1268aijkl |
| 1268b | 1268bijl |
| 1268bk | 1268bijkl |
| 1268k | 1268ijk |
| 1268kl | 1268ijkl |
| 1268l | 1268ijl |
| 126a | 126aijk |
| 126ab | 126abijkl |

| | |
|--------|------------------|
| 126abc | 12345678abcdijkl |
| 126ac | 123568acijk |
| 126acl | 12345678acdijkl |
| 126al | 126aijkl |
| 126b | 126bijl |
| 126bc | 12345678bcdijkl |
| 126bk | 126bijkl |
| 126c | 123568cijk |
| 126cl | 12345678cdijkl |
| taha | tetha-bar |
| 126k | 126ijk |
| 126kl | 126ijkl |
| 126l | 126ijl |
| 127 | 127ij |
| 1278 | 1278ij |
| 127a | 1278aijk |
| 127ab | 1278abijkl |
| 127ac | 124578acijk |
| 127al | 1278aijkl |
| 127b | 127bijl |
| 127bk | 1278bijkl |
| 127c | 124578cijk |
| 127k | 1278ijk |
| 127kl | 1278ijkl |
| 127l | 127ijl |
| 128 | 128ij |
| 128a | 128aijk |
| 128ab | 128abijkl |
| 128al | 128aijkl |
| 128b | 128bijl |
| 128bd | 123678bdijl |
| 128bk | 128bijkl |
| 128d | 123678dijl |
| 128k | 128ijk |
| 128kl | 128ijkl |
| 128l | 128ijl |
| 12a | 12aijk |
| 12ab | 12abijkl |
| 12abc | 1258abcijkl |
| 12abcd | 12345678abcdijkl |
| 12abd | 123678abdijkl |
| 12ac | 1258acijk |
| 12acd | 12345678acdijkl |
| 12acl | 1258acijkl |
| 12ad | 123678adijkl |
| 12al | 12aijkl |
| 12b | 12bijl |
| 12bc | 1258bcijkl |
| 12bcd | 12345678bcdijkl |

| | |
|---------|------------------|
| 12bd | 1267bdijl |
| 12bdk | 123678bdijkl |
| 12bk | 12bijkl |
| 12c | 1258cijk |
| 12cd | 12345678cdijkl |
| 12cl | 1258cijkl |
| 12d | 1267dijl |
| 12dk | 123678dijkl |
| 12k | 12ijk |
| 12kl | 12ijkl |
| 12l | 12ijl |
| 13 | 13 |
| 135 | 135ij |
| 1357 | 1357ij |
| 1357a | 1357aijk |
| 1357ab | 1357abijkl |
| 1357ac | 12345678acijk |
| 1357al | 1357aijkl |
| 1357c | 12345678cijk |
| 1357k | 1357ijk |
| 1357kl | 1357ijkl |
| 135a | 135aijk |
| 135ab | 135abijkl |
| 135abc | 12345678abcdijkl |
| 135ac | 123568acijk |
| 135acl | 12345678acdijkl |
| 135al | 135aijkl |
| 135b | 135bijl |
| 135bc | 12345678bcdijkl |
| 135bd | 134568bdijl |
| 135bk | 135bijkl |
| 135c | 123568cijk |
| 135cl | 12345678cdijkl |
| 135d | 134568dijl |
| 135k | 135ijk |
| 135kl | 135ijkl |
| tetha | tetha-bar |
| 135l | 135ijl |
| 136 | 136 |
| 1368 | 1368 |
| 1368a | 1368ak |
| 1368ab | 1368abkl |
| 1368abi | 1368abikl |
| 1368abj | 1368abdijkl |
| 1368ai | 1368aik |
| 1368ail | 1368aikl |
| 1368aj | 1368aijk |
| 1368ajl | 1368aijkl |
| 1368al | 1368akl |

| | |
|---------|--------------|
| 1368b | 1368b1 |
| 1368bi | 1368bil |
| 1368bik | 1368bikl |
| 1368bj | 1368bdijl |
| 1368bjk | 1368bdijkl |
| 1368bk | 1368bkl |
| 1368i | 1368i |
| 1368ik | 1368ik |
| 1368ikl | 1368ikl |
| 1368il | 1368il |
| 1368j | 1368ij |
| 1368jk | 1368ijk |
| 1368jkl | 1368ijkl |
| 1368jl | 1368ijl |
| 1368k | 1368k |
| 1368kl | 1368kl |
| 1368l | 1368l |
| 136a | 136ak |
| 136ab | 136abkl |
| 136abi | 136abikl |
| 136abj | 136abijkl |
| 136ai | 136aik |
| 136ail | 136aikl |
| 136aj | 136aijk |
| 136ajl | 136aijkl |
| 136al | 136akl |
| 136b | 136b1 |
| 136bi | 136bil |
| 136bik | 136bikl |
| 136bj | 136bijl |
| 136bjk | 136bijkl |
| 136bk | 136bkl |
| 136i | 136i |
| 136ik | 136ik |
| 136ikl | 136ikl |
| 136il | 136il |
| 136j | 136ij |
| 136jk | 136ijk |
| 136jkl | 136ijkl |
| 136jl | 136ijl |
| 136k | 136k |
| 136kl | 136kl |
| 136l | 136l |
| 13a | 13ak |
| 13ab | 13abkl |
| 13abc | 1368abc1 |
| 13abci | 1368abcikl |
| 13abcj | 1368abcdijkl |
| 13abd | 1368abd1 |

| | |
|--------|-------------|
| 13abdi | 1368abdikl |
| 13abdj | 1368abdijkl |
| 13abi | 13abikl |
| 13abj | 13abijkl |
| 13ac | 1368ack |
| 13aci | 1368acik |
| 13acil | 1368acikl |
| 13acj | 1368acijk |
| 13acjl | 1368acijkl |
| 13acl | 1368ackl |
| 13ad | 1368adkl |
| 13adi | 1368adikl |
| 13adj | 1368adijkl |
| 13ai | 13aik |
| tetha | tetha-bar |
| 13ail | 13aikl |
| 13aj | 13aijk |
| 13ajl | 13aijkl |
| 13al | 13akl |
| 13b | 13bl |
| 13bc | 1368bckl |
| 13bci | 1368bcikl |
| 13bcj | 1368bcdijkl |
| 13bd | 1368bdl |
| 13bdi | 1368bdil |
| 13bdik | 1368bdikl |
| 13bdj | 1368bdijl |
| 13bdjk | 1368bdijkl |
| 13bdk | 1368bdkl |
| 13bi | 13bil |
| 13bik | 13bikl |
| 13bj | 13bijl |
| 13bjk | 13bijkl |
| 13bk | 13bkl |
| 13c | 1368ck |
| 13ci | 1368cik |
| 13cil | 1368cikl |
| 13cj | 1368cijk |
| 13cjl | 1368cijkl |
| 13cl | 1368ckl |
| 13d | 1368dl |
| 13di | 1368dil |
| 13dik | 1368dikl |
| 13dj | 1368dijl |
| 13djk | 1368diijkl |
| 13dk | 1368dkl |
| 13i | 13i |
| 13ik | 13ik |
| 13ikl | 13ikl |

| | |
|--------|------------------|
| 13il | 13il |
| 13j | 13ij |
| 13jk | 13ijk |
| 13jkl | 13ijkl |
| 13jl | 13ijl |
| 13k | 13k |
| 13kl | 13kl |
| 13l | 13l |
| 15 | 15ij |
| 15a | 15aijk |
| 15ab | 15abijkl |
| 15abc | 1258abcijkl |
| 15abcd | 12345678abcdijkl |
| 15ac | 1258acijk |
| 15acd | 12345678acdijkl |
| 15acl | 1258acijkl |
| 15ad | 1456adijkl |
| 15al | 15aijkl |
| 15c | 1258cijk |
| 15cd | 12345678cdijkl |
| 15cl | 1258cijkl |
| 15k | 15ijk |
| 15kl | 15ijkl |
| 16 | 16 |
| 16a | 16ak |
| 16ab | 16abkl |
| 16abc | 1368abckl |
| 16abci | 1368abcikl |
| 16abcj | 1368abcdijkl |
| 16abi | 16abikl |
| 16abj | 16abijkl |
| 16ac | 1368ack |
| 16aci | 1368acik |
| 16acil | 1368acikl |
| 16acj | 1368acijk |
| 16acjl | 1368acijkl |
| 16acl | 1368ackl |
| 16ai | 16aik |
| 16ail | 16aikl |
| 16aj | 16aijk |
| 16ajl | 16aijkl |
| tetha | tetha-bar |
| 16al | 16akl |
| 16b | 16bl |
| 16bc | 1368bckl |
| 16bci | 1368bcikl |
| 16bcj | 1368bcdijkl |
| 16bi | 16bil |
| 16bik | 16bikl |

| | |
|--------|------------------|
| 16bj | 16bijl |
| 16bjk | 16bijkl |
| 16bk | 16bkl |
| 16c | 1368ck |
| 16ci | 1368cik |
| 16cil | 1368cikl |
| 16cj | 1368cijk |
| 16cjl | 1368cijkl |
| 16cl | 1368ckl |
| 16i | 16i |
| 16ik | 16ik |
| 16ikl | 16ikl |
| 16il | 16il |
| 16j | 16ij |
| 16jk | 16ijk |
| 16jkl | 16ijkl |
| 16jl | 16ijl |
| 16k | 16k |
| 16kl | 16kl |
| 16l | 16l |
| 17 | 17 |
| 17a | 17ak |
| 17ab | 17abkl |
| 17abc | 124578abcijkl |
| 17abcd | 12345678abcdijkl |
| 17abi | 17abijkl |
| 17ac | 1478acijk |
| 17acd | 12345678acdijkl |
| 17acl | 124578acijkl |
| 17ad | 123678adijkl |
| 17ai | 17aijk |
| 17ail | 17aijkl |
| 17aj | 17aijk |
| 17ajl | 17aijkl |
| 17al | 17akl |
| 17c | 1478cijk |
| 17cd | 12345678cdijkl |
| 17cl | 124578cijkl |
| 17i | 17ij |
| 17ik | 17ijk |
| 17ikl | 17ijkl |
| 17il | 17ijl |
| 17k | 17k |
| 17kl | 17kl |
| 18 | 18 |
| 18a | 18ak |
| 18ab | 18abkl |
| 18abd | 1368abdkl |
| 18abdi | 1368abdikl |

| | |
|--------|--------------|
| 18abdj | 1368abdijkl |
| 18abi | 18abikl |
| 18abj | 18abijkl |
| 18ad | 1368adkl |
| 18adi | 1368adikl |
| 18adj | 1368adijkl |
| 18ai | 18aik |
| 18ail | 18aikl |
| 18aj | 18aijk |
| 18ajl | 18aijkl |
| 18al | 18akl |
| 18b | 18bl |
| 18bd | 1368bdl |
| 18bdi | 1368bdil |
| 18bdik | 1368bdikl |
| 18bdj | 1368bdijl |
| 18bdjk | 1368bdijkl |
| 18bdk | 1368bdkl |
| 18bi | 18bil |
| tetha | tetha-bar |
| 18bik | 18bikl |
| 18bj | 18bijl |
| 18bjk | 18bijkl |
| 18bk | 18bkl |
| 18d | 1368dl |
| 18di | 1368dil |
| 18dik | 1368dikl |
| 18dj | 1368dijl |
| 18djkl | 1368dijkl |
| 18dk | 1368dkl |
| 18i | 18i |
| 18ik | 18ik |
| 18ikl | 18ikl |
| 18il | 18il |
| 18j | 18ij |
| 18jk | 18ijk |
| 18jkl | 18ijkl |
| 18jl | 18ijl |
| 18k | 18k |
| 18kl | 18kl |
| 18l | 18l |
| 1a | 1ak |
| 1ab | 1abkl |
| 1abc | 18abckl |
| 1abcd | 1368abcdkl |
| 1abcdi | 1368abcdikl |
| 1abcdj | 1368abcdijkl |
| 1abci | 18abcikl |
| 1abcj | 18abcijkl |

| | |
|-------|-------------|
| 1abd | 16abdkl |
| 1abdi | 16abdikl |
| 1abdj | 16abdijkl |
| 1abi | 1abikl |
| 1abj | 1abijkl |
| 1ac | 18ack |
| 1acd | 1368acdkl |
| 1acdi | 1368acdikl |
| 1acdj | 1368acdijkl |
| 1aci | 18acik |
| 1acil | 18acikl |
| 1acj | 18acijk |
| 1acjl | 18acijkl |
| 1acl | 18ackl |
| 1ad | 16adkl |
| 1adi | 16adikl |
| 1adj | 16adijkl |
| 1ai | 1aik |
| 1ail | 1aikl |
| 1aj | 1aijk |
| 1ajl | 1aijkl |
| 1al | 1akl |
| 1b | 1bl |
| 1bc | 18bckl |
| 1bcd | 1368bcdkl |
| 1bcdi | 1368bcdikl |
| 1bcdj | 1368bcdijkl |
| 1bci | 18bcikl |
| 1bcj | 18bcijk |
| 1bd | 16bdl |
| 1bdi | 16bdil |
| 1bdik | 16bdikl |
| 1bdj | 16bdijl |
| 1bdjk | 16bdijkl |
| 1bdk | 16bdkl |
| 1bi | 1bil |
| 1bik | 1bikl |
| 1bj | 1bijl |
| 1bjk | 1bijkl |
| 1bk | 1bkl |
| 1c | 18ck |
| 1cd | 1368cdkl |
| 1cdi | 1368cdikl |
| 1cdj | 1368cdijkl |
| 1ci | 18cik |
| 1cil | 18cikl |
| tetha | tetha-bar |
| 1cj | 18cijk |
| 1cjl | 18cijkl |

| | |
|--------|----------|
| 1cl | 18ckl |
| 1d | 16dl |
| 1di | 16dil |
| 1dik | 16dikl |
| 1dj | 16dijl |
| 1djk | 16dijkl |
| 1dk | 16dkl |
| 1i | 1i |
| 1ik | 1ik |
| 1ikl | 1ikl |
| 1il | 1il |
| 1j | 1ij |
| 1jk | 1ijk |
| 1jkl | 1ijkl |
| 1jl | 1ijl |
| 1k | 1k |
| 1kl | 1kl |
| 1l | 1l |
| a | ak |
| ab | abkl |
| abc | abckl |
| abcd | abcdkl |
| abcdi | abcdikl |
| abcdij | abcdijkl |
| abci | abcikl |
| abcij | abcijkl |
| abcj | abcjkl |
| abi | abikl |
| abij | abijkl |
| ac | ack |
| acd | acdkl |
| acdi | acdikl |
| acdij | acdijkl |
| acdj | acdjkl |
| aci | acik |
| acij | acijk |
| acijl | acijkl |
| acil | acikl |
| acj | acjk |
| acjl | acjkl |
| acl | ackl |
| ad | adkl |
| adi | adikl |
| adij | adijkl |
| adj | adjkl |
| ai | aik |
| aij | aijk |
| aijl | aijkl |
| ail | aikl |

| | |
|------|--------|
| aj | ajk |
| ajl | ajkl |
| al | akl |
| c | ck |
| cd | cdkl |
| cdi | cdikl |
| cdij | cdijkl |
| ci | cik |
| cij | cijk |
| cijl | cijkl |
| cil | cikl |
| cj | cjk |
| cjl | cjkl |
| cl | ckl |
| i | i |
| ij | ij |
| ijk | ijk |
| ijkl | ijkl |
| ik | ik |
| ikl | ikl |
| il | il |
| k | k |
| kl | kl |

nr. lines 599.000000

Table 3

References

- [1] J.A. Kalman, On the Postulates for Lattices, Math. Annalen, Bd. 137. S.362-370 (1959).
- [2] G. Laslo, The Classification of Noncommutative Lattices according to Absorption laws, Proceeding of the Annual Meeting of the Romanian Society of Mathematical Sciences, 1997, 29 May-June 1, Tome 1, 77-84.
- [3] G. Laslo and I. Cozac, On Noncommutative Generalizations of Lattices, Mathematica, Tome 40 (63) nr. 1 (1998), 186-196.
- [4] G. Laslo and J.E. Leech, Green's Equivalence on Noncommutative Lattices, Acta Scientiarum Mathematicarum, Bolyai Institute, University of Szeged, Volume 68 (2002), 501-533.
- [5] Sorokin, Yu.I., Independent systems of axioms defining a lattice, Ukrain. mat. z, 3, 85-97 (1951) (russ).