

Some interpolation properties and tensor product stability of Stolz Mappings

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1 Introduction

Let X be a Banach space and let $T \in \mathbf{L}(X)$ be a linear and bounded operator $T : X \rightarrow X$.

By $\{a_n(T)\}$ and $\{e_n(T)\}$ we denote the sequences of the approximation numbers and entropy numbers of T (dyadic entropy numbers) [2], [5], [8].

The class of Stolz mappings has been defined by K. Iseki, see [4], as follows:

$$L_{STOL,p}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}, 0 < p < \infty$$

where $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$.

In [4] is proved that if $\lim_{n \rightarrow \infty} \alpha_n \neq 0$ then

$$L_{STOL,p}(X) = L_p(X) = \left\{ T : \left(\sum_{n=1}^{\infty} a_n^p(T) \right)^{\frac{1}{p}} < \infty \right\}$$

Remark 1.1. The application $\Phi_n^\alpha : \{a_i(T)\}_1^n \rightarrow \sum_{i=1}^n \alpha_i a_i(T)$ is a symmetric norming function, which is not equivalent with the maximal function $\Phi_1 : \{a_i(T)\}_1^n \rightarrow \sum_{i=1}^n a_i(T)$ if $\lim_{n \rightarrow \infty} a_n(T) = 0$ [2], [8]. In this way we remark that the class of Stolz mappings is a particular case of the classes

$$L_{M_\Phi,p}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{\Phi(\{a_i(T)\}_1^n)}{\Phi(n)} \right)^p \right)^{\frac{1}{p}} < \infty \right\}, 0 < p < \infty$$

where $\Phi(n) = \Phi(\underbrace{1, \dots, 1}_n, 0, 0, \dots)$ and Φ is a symmetric norming function. The classes $L_{M_\Phi,p}$ has been presented in [6] and in the other lectures.

For the properties of the function Φ and $\{a_n(T)\}$, $\{e_n(T)\}$ it can see [2], [8].

It is known that $L_{STOL,p}$ is an operator ideal (quasinormed). Also $L_{M_\Phi,p}$ is a quasinormed operator ideal, because

$$\sum_{n=1}^k a_n(S+T) \leq 2 \sum_{n=1}^k [a_n(S) + a_n(T)], k = 1, 2, \dots$$

[8], [9]. In the following we present some properties for $L_{STOL,p}$.

2 Results

Theorem 2.1. *If $S \in L_{STOL,s,q}(X)$ and $T \in L_{STOL,t,r}(X)$, then $ST \in L_{STOL,p}(X)$, where $1 = \frac{1}{s} + \frac{1}{t}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, $1 \leq p < \infty$ and*

$$L_{STOL,s,q}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i^s(T) \right)^{\frac{q}{s}} \right)^{\frac{1}{q}} < \infty \right\}$$

Proof.

$$\begin{aligned} \|ST\|_{STOL,p} &= \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i(ST) \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^\alpha(\{a_i(ST)\}_1^n)}{\Phi^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

where $\Phi^\alpha(n) = \Phi^\alpha(\underbrace{1, \dots, 1}_n, 0, 0, \dots)$. Since

$$\sum_{i=1}^k a_i(ST) \leq 2 \sum_{i=1}^k [a_i(S) \cdot a_i(T)], \quad k = 1, 2, \dots$$

and Φ^α is a symmetric norming function it follows:

$$\begin{aligned} \|ST\|_{STOL,p} &\leq 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi^\alpha(\{a_i(S)a_i(T)\}_{i=1}^n)}{\Phi^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \\ &\leq 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi_{(s)}^\alpha(\{a_i(S)\}_{i=1}^n) \cdot \Phi_{(t)}^\alpha(\{a_i(T)\}_{i=1}^n)}{\Phi_{(s)}^\alpha(n) \cdot \Phi_{(t)}^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \\ &\leq 2 \left(\sum_{n=1}^{\infty} \left(\frac{\Phi_{(s)}^\alpha(\{a_i(S)\}_{i=1}^n)}{\Phi_{(s)}^\alpha(n)} \right)^q \right)^{\frac{1}{q}} \cdot \left(\sum_{n=1}^{\infty} \left(\frac{\Phi_{(t)}^\alpha(\{a_i(T)\}_{i=1}^n)}{\Phi_{(t)}^\alpha(n)} \right)^r \right)^{\frac{1}{r}} \end{aligned}$$

where $1 = \frac{1}{s} + \frac{1}{t}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Hence $ST \in L_{STOL,p}(X)$. ■

Theorem 2.2. *The classes $L_{STOL,p}(X)$ are tensor product stable for all tensor norms, if the sequence $(\alpha_n)_n$ is such that $\alpha_{n^2} \leq \frac{C}{n} \alpha_n$, $(\forall) n = 1, 2, \dots$ and C is a constant (depending only of the sequence $\alpha = (\alpha_1, \alpha_2, \dots)$).*

Proof. The proof is a corollary of the inequality:

$$\sum_{n=1}^k \alpha_n a_n(S \otimes T) \leq C(\alpha) \sum_{n=1}^k \alpha_n [a_n(S) + a_n(T)]$$

¹We remark that $\Phi_{(s)}^\alpha(a_i(S)) = \left(\sum_{i=1}^n \alpha_i a_i^s(S) \right)^{\frac{1}{s}}$

[7], which is true for all (fixed!) $S, T \in \mathbf{L}(X)$.

We obtain:

$$\begin{aligned} \|S \otimes T\|_{STOL,p} &\leq C(\alpha) \cdot \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i [a_i(S) + a_i(T)] \right)^p \right)^{\frac{1}{p}} \\ &\leq C(\alpha, p) \left[\|S\|_{STOL,p} + \|T\|_{STOL,p} \right] < \infty \end{aligned}$$

■ Now we present a interpolation theorem of Riesz-Thorin type for the ideals

$$L_{STOL,p}^{(e)}(X) = \left\{ T : \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}$$

where $\{e_i(T)\}$ is the sequence of the dyadic entropy numbers.

If (X_0, X_1) is an interpolation couple of two normed spaces and $(X_0, X_1)_{\theta,q}$ is the interpolation space, $\theta \in (0, 1)$, $0 < q < \infty$, it is known that:

$$e_{2n-1} \left(T : (X_0, X_1)_{\theta,q} \rightarrow X \right) \leq 2e_n(T : X_0 \rightarrow X)^{1-\theta} \cdot e_n(T : X_1 \rightarrow X)^\theta$$

for all normed spaces X .

Since

$$\|T\|_{STOL,p}^{(e)} \sim \|T\|_{STOL,p}^{*(e)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_{2i-1}(T) \right)^p \right)^{\frac{1}{p}}$$

we obtain

$$\begin{aligned} &\left\| T : (X_0, X_1)_{\theta,q} \rightarrow X \right\|_{STOL,p}^{(e)} \leq \\ &\leq C \left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{1-\theta} \cdot e_i(T : X_1 \rightarrow X)^\theta \right)^p \right)^{\frac{1}{p}} \\ &\leq C \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_1 + \dots + \alpha_n} \left(\sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{p_1} \right)^{\frac{1-\theta}{p_1}} \left(\sum_{i=1}^n \alpha_i e_i(T : X_1 \rightarrow X)^{p_2} \right)^{\frac{\theta}{p_2}} \right) \\ &\leq C \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \dots + \alpha_n} \left(\sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{p_1} \right)^{\frac{r}{p_1}} \right]^{\frac{1-\theta}{r}} \cdot \\ &\quad \cdot \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \dots + \alpha_n} \left(\sum_{i=1}^n \alpha_i e_i(T : X_1 \rightarrow X)^{p_2} \right)^{\frac{s}{p_2}} \right]^{\frac{\theta}{s}} \end{aligned}$$

where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\theta \in (0, 1)$, $0 < p_1 < p_2 < \infty$ and $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$.

Hence we obtain:

Proposition 2.3. $L_{STOL,p_1,r}^{(e)}(X_0, X) \cap L_{STOL,p_2,s}^{(e)}(X_1, X) \subseteq L_{STOL,p}^{(e)}((X_0, X_1)_{\theta,q}, X)$
if $0 < p_1 < p_2 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$, $\theta \in (0, 1)$, $0 < r < s < \infty$.

Remark 2.4. All results are valid if the function Φ^α is replaced by an other function Φ .

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