

# Some interpolation properties and tensor product stability of Stolz Mappings

Nicolae Tița

## 1 Introduction

Let  $X$  be a Banach space and let  $T \in \mathbf{L}(X)$  be a linear and bounded operator  $T : X \rightarrow X$ .

By  $\{a_n(T)\}$  and  $\{e_n(T)\}$  we denote the sequences of the approximation numbers and entropy numbers of  $T$  (dyadic entropy numbers) [2], [5], [8].

The class of Stolz mappings has been defined by K. Iseki, see [4], as follows:

$$L_{STOL,p}(X) = \left\{ T : \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 0 < p < \infty$$

where  $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ .

In [4] is proved that if  $\lim_{n \rightarrow \infty} \alpha_n \neq 0$  then

$$L_{STOL,p}(X) = L_p(X) = \left\{ T : \left( \sum_{n=1}^{\infty} a_n^p(T) \right)^{\frac{1}{p}} < \infty \right\}$$

*Remark 1.1.* The application  $\Phi_n^\alpha : \{a_i(T)\}_1^n \rightarrow \sum_{i=1}^n \alpha_i a_i(T)$  is a symmetric norming function, which is not equivalent with the maximal function  $\Phi_1 : \{a_i(T)\}_1^n \rightarrow \sum_{i=1}^n a_i(T)$  if  $\lim_{n \rightarrow \infty} a_n(T) = 0$  [2], [8]. In this way we remark that the class of Stolz mappings is a particular case of the classes

$$L_{M_\Phi,p}(X) = \left\{ T : \left( \sum_{n=1}^{\infty} \left( \frac{\Phi(\{a_i(T)\}_1^n)}{\Phi(n)} \right)^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 0 < p < \infty$$

where  $\Phi(n) = \Phi(\underbrace{1, \dots, 1}_n, 0, 0, \dots)$  and  $\Phi$  is a symmetric norming function. The classes  $L_{M_\Phi,p}$  has been presented in [6] and in the other lectures.

For the properties of the function  $\Phi$  and  $\{a_n(T)\}$ ,  $\{e_n(T)\}$  it can see [2], [8].

It is known that  $L_{STOL,p}$  is an operator ideal (quasinormed). Also  $L_{M_\Phi,p}$  is a quasinormed operator ideal, because

$$\sum_{n=1}^k a_n(S+T) \leq 2 \sum_{n=1}^k [a_n(S) + a_n(T)], \quad k = 1, 2, \dots$$

[8], [9]. In the following we present some properties for  $L_{STOL,p}$ .

## 2 Results

**Theorem 2.1.** *If  $S \in L_{STOL,s,q}(X)$  and  $T \in L_{STOL,t,r}(X)$ , then  $ST \in L_{STOL,p}(X)$ , where  $1 = \frac{1}{s} + \frac{1}{t}$ ,  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ ,  $1 \leq p < \infty$  and*

$$L_{STOL,s,q}(X) = \left\{ T : \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i^s(T) \right)^{\frac{q}{s}} \right)^{\frac{1}{q}} < \infty \right\}$$

**Proof.**

$$\begin{aligned} \|ST\|_{STOL,p} &= \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i a_i(ST) \right)^p \right)^{\frac{1}{p}} \\ &= \left( \sum_{n=1}^{\infty} \left( \frac{\Phi^\alpha(\{a_i(ST)\}_1^n)}{\Phi^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

where  $\Phi^\alpha(n) = \Phi^\alpha(\underbrace{1, \dots, 1}_n, 0, 0, \dots)$ . Since

$$\sum_{i=1}^k a_i(ST) \leq 2 \sum_{i=1}^k [a_i(S) \cdot a_i(T)], \quad k = 1, 2, \dots$$

and  $\Phi^\alpha$  is a symmetric norming function it follows:

$$\begin{aligned} \|ST\|_{STOL,p} &\leq 2 \left( \sum_{n=1}^{\infty} \left( \frac{\Phi^\alpha(\{a_i(S)a_i(T)\}_{i=1}^n)}{\Phi^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \\ &\leq 2 \left( \sum_{n=1}^{\infty} \left( \frac{\Phi_{(s)}^\alpha(\{a_i(S)\}_{i=1}^n) \cdot \Phi_{(t)}^\alpha(\{a_i(T)\}_{i=1}^n)}{\Phi_{(s)}^\alpha(n) \cdot \Phi_{(t)}^\alpha(n)} \right)^p \right)^{\frac{1}{p}} \\ &\leq 2 \left( \sum_{n=1}^{\infty} \left( \frac{\Phi_{(s)}^\alpha(\{a_i(S)\}_{i=1}^n)}{\Phi_{(s)}^\alpha(n)} \right)^q \right)^{\frac{1}{q}} \cdot \left( \sum_{n=1}^{\infty} \left( \frac{\Phi_{(t)}^\alpha(\{a_i(T)\}_{i=1}^n)}{\Phi_{(t)}^\alpha(n)} \right)^r \right)^{\frac{1}{r}} \end{aligned}$$

where  $1 = \frac{1}{s} + \frac{1}{t}$ ,  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ . Hence  $ST \in L_{STOL,p}(X)$ . ■

**Theorem 2.2.** *The classes  $L_{STOL,p}(X)$  are tensor product stable for all tensor norms, if the sequence  $(\alpha_n)_n$  is such that  $\alpha_{n^2} \leq \frac{C}{n} \alpha_n$ ,  $(\forall)n = 1, 2, \dots$  and  $C$  is a constant (depending only of the sequence  $\alpha = (\alpha_1, \alpha_2, \dots)$ ).*

**Proof.** The proof is a corollary of the inequality:

$$\sum_{n=1}^k \alpha_n a_n(S \otimes T) \leq C(\alpha) \sum_{n=1}^k \alpha_n [a_n(S) + a_n(T)]$$

<sup>1</sup>We remark that  $\Phi_{(s)}^\alpha(a_i(S)) = \left( \sum_{i=1}^n \alpha_i a_i^s(S) \right)^{\frac{1}{s}}$

[7], which is true for all (fixed!)  $S, T \in \mathbf{L}(X)$ .

We obtain:

$$\begin{aligned} \|S \otimes T\|_{STOL,p} &\leq C(\alpha) \cdot \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i [a_i(S) + a_i(T)] \right)^p \right)^{\frac{1}{p}} \\ &\leq C(\alpha, p) \left[ \|S\|_{STOL,p} + \|T\|_{STOL,p} \right] < \infty \end{aligned}$$

■ Now we present a interpolation theorem of Riesz-Thorin type for the ideals

$$L_{STOL,p}^{(e)}(X) = \left\{ T : \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_i(T) \right)^p \right)^{\frac{1}{p}} < \infty \right\}$$

where  $\{e_i(T)\}$  is the sequence of the dyadic entropy numbers.

If  $(X_0, X_1)$  is an interpolation couple of two normed spaces and  $(X_0, X_1)_{\theta,q}$  is the interpolation space,  $\theta \in (0, 1)$ ,  $0 < q < \infty$ , it is known that:

$$e_{2n-1} \left( T : (X_0, X_1)_{\theta,q} \rightarrow X \right) \leq 2e_n(T : X_0 \rightarrow X)^{1-\theta} \cdot e_n(T : X_1 \rightarrow X)^\theta$$

for all normed spaces  $X$ .

Since

$$\|T\|_{STOL,p}^{(e)} \sim \|T\|_{STOL,p}^{*(e)} = \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_{2i-1}(T) \right)^p \right)^{\frac{1}{p}}$$

we obtain

$$\begin{aligned} &\left\| T : (X_0, X_1)_{\theta,q} \rightarrow X \right\|_{STOL,p}^{(e)} \leq \\ &\leq C \left( \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{1-\theta} \cdot e_i(T : X_1 \rightarrow X)^\theta \right)^p \right)^{\frac{1}{p}} \\ &\leq C \sum_{n=1}^{\infty} \left( \frac{1}{\alpha_1 + \dots + \alpha_n} \left( \sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{p_1} \right)^{\frac{1-\theta}{p_1}} \left( \sum_{i=1}^n \alpha_i e_i(T : X_1 \rightarrow X)^{p_2} \right)^{\frac{\theta}{p_2}} \right) \\ &\leq C \left[ \sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \dots + \alpha_n} \left( \sum_{i=1}^n \alpha_i e_i(T : X_0 \rightarrow X)^{p_1} \right)^{\frac{r}{p_1}} \right]^{\frac{1-\theta}{r}} \cdot \\ &\quad \cdot \left[ \sum_{n=1}^{\infty} \frac{1}{\alpha_1 + \dots + \alpha_n} \left( \sum_{i=1}^n \alpha_i e_i(T : X_1 \rightarrow X)^{p_2} \right)^{\frac{s}{p_2}} \right]^{\frac{\theta}{s}} \end{aligned}$$

where  $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ ,  $\theta \in (0, 1)$ ,  $0 < p_1 < p_2 < \infty$  and  $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$ .

Hence we obtain:

**Proposition 2.3.**  $L_{STOL,p_1,r}^{(e)}(X_0, X) \cap L_{STOL,p_2,s}^{(e)}(X_1, X) \subseteq L_{STOL,p}^{(e)}((X_0, X_1)_{\theta,q}, X)$   
if  $0 < p_1 < p_2 < \infty$ ,  $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ ,  $1 = \frac{1-\theta}{r} + \frac{\theta}{s}$ ,  $\theta \in (0, 1)$ ,  $0 < r < s < \infty$ .

*Remark 2.4.* All results are valid if the function  $\Phi^\alpha$  is replaced by an other function  $\Phi$ .

## References

- [1] J. Berg, J. Löfstrom *Interpolation spaces*, Springer-Verlag, 1976
- [2] I. Gohberg, M. Krein *Introducere în teoria operatorilor liniari neautoadjuncți în spații hilbert (lb. rusă)*, Nauka-Moscova, 1965
- [3] J. Peetre, G. Sparr *Interpolation of normed Abelian groups*, Ann. Mat. Pura et Appl. 92(1972), 216-262
- [4] N. Tița *On Stoltz Mappings*, Math. Japonica 4, 26 (1981) 495 - 496
- [5] N. Tița *Ideale de entropie și ideale de s-numere*, Studii Cercetări Matematice, 43 (1991), 77-81
- [6] N. Tița *Lecture notes in "Operator Theory"*, Universitatea "Transilvania" Brașov, 1993
- [7] N. Tița *Some inequalities for the approximation numbers of tensor product operators*, An st. Univ. Iași 40 (1994) 329 - 331
- [8] N. Tița *Ideale generate de s-numere*, Universitatea "Transilvania" Brașov, 1998
- [9] M. Talpău Dimitriu *Aplicații la unele ideale de s-numere*, Editura Matrix Rom, 2007