The operatorial form of the overdetermined infinite linear systems

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Proceedings of the European Integration between Tradition and Modernity
"Petru Maior" University of Târgu-Mureş
Oktober, 22-23, 2009

Let X_1 and X_2 be two real or complex Hilbert spaces, respectively. We consider the linear and continuous operator $T : X_1 \to X_2$, where $T^* : X_2 \to X_1$ is the adjoint operator of T. Let us take the equation $T(x) = b$, where $x \in X_1$ is the unknown and $b \in X_2$ is a fixed element.

Theorem 1. *If $x^* \in X_1$ verifies the condition $(T(x^*)-b) \in \operatorname{Ker}(T^*)$ then $\|T(x^*)-b\| \leq \|T(x) - b\|$ for all $x \in X_1$.*

Proof. We have the following:

$$\begin{aligned}
\|b - T(x)\|^2 &= \|b - T(x^*) + T(x^* - x)\|^2 = \\
&= (b - T(x^*) + T(x^* - x), b - T(x^*) + T(x^* - x)) = \\
&= (b - T(x^*), b - T(x^*)) + (b - T(x^*), T(x^* - x)) + \\
&\quad +(T(x^* - x), b - T(x^*)) + (T(x^* - x), T(x^* - x)) = \\
&= \|b - T(x^*)\|^2 + (T^*(b - T(x^*)), x^* - x) + \\
&\quad (x^* - x, T^*(b - T(x^*))) + \|T(x^* - x)\|^2 = \\
&= \|b - T(x^*)\|^2 + \|T(x^* - x)\|^2 \geq \|b - T(x^*)\|^2.
\end{aligned}$$

□

Next we give some applications.

Application 1. We consider the real, finite, overdetermined linear system:

$$\begin{cases}
a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n = b_0 \\
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{cases}$$

where $m > n$ and $a_{ij}, b_i \in \mathbb{R}$ for all $i = \overline{0,m}$ and $j = \overline{1,n}$. Let $A = (a_{ij})_{\substack{i=\overline{0,m} \\ j=\overline{1,n}}}$ be the matrix of the real linear system and $b = (b_i)_{i=\overline{0,m}}$ is the constant term. Then we obtain the following linear and continuous operator $T : \mathbb{R}^n \to \mathbb{R}^{m+1}$, and $T(x) = A \cdot x$, for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. So the adjoint operator T^* has the matrix A^T, which is the transpose of the matrix A. Using our theorem, from the condition $(T(x^*) - b) \in \operatorname{Ker}(T^*)$ we obtain $T^*(T(x^*) - b) = 0$, i.e. $T^*(T(x^*)) = T^*(b)$, which has the equivalent matrix form $A^T \cdot A \cdot x^* = A^T \cdot b$. So from our theorem we reobtain the following well known result: if $A^T \cdot A \cdot x^* = A^T \cdot b$ then $\|A \cdot x^* - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{R}^n$ (see [1], [2] or [3]).

Application 2. We consider the real, infinite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\ \vdots \end{cases}$$

where $a_{ij}, b_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and $j = \overline{1,n}$. Let $a_j = (a_{ij})_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ and $b = (b_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$, so $A = (a_1 a_2 \ldots a_n)$ is the matrix of the infinite linear system and b is the constant term. Then we obtain the following linear and continuous operator $T : \mathbb{R}^n \to l^2(\mathbb{R})$, $T(x) = A \cdot x$, for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. So the adjoint operator T^* has the matrix A^T, which is the transpose of the matrix A. Using our theorem, from the condition $(T(x^*) - b) \in \operatorname{Ker}(T^*)$ we obtain $T^*(T(x^*) - b) = 0$, i.e. $T^*(T(x^*)) = T^*(b)$, which has the equivalent matrix form $A^T \cdot A \cdot x^* = A^T \cdot b$. So from our theorem we reobtain the following result: if $A^T \cdot A \cdot x^* = A^T \cdot b$ then $\|A \cdot x^* - b\| \le \|A \cdot b - b\|$ for all $x \in \mathbb{R}^n$ (see [4]).

Application 3. We consider the complex, finite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

where $m > n$ and $a_{ij}, b_i \in \mathbb{C}$ for all $i = \overline{0,m}$ and $j = \overline{1,n}$. Let $A = (a_{ij})_{\substack{i=\overline{0,m} \\ j=\overline{1,n}}}$ be the matrix of the complex linear system and $b = (b_i)_{i=\overline{0,m}}$ is the constant term. Then we obtain the following linear and continuous operator $T : \mathbb{C}^n \to \mathbb{C}^{m+1}$, $T(x) = A \cdot x$, for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$. So the adjoint operator T^* has the matrix $\overline{A}^T$, which is the transpose of the matrix A and taking the complex conjugate for all elements of A^T. Using our theorem, from the condition $(T(x^*) - b) \in \operatorname{Ker}(T^*)$ we obtain $T^*(T(x^*) - b) = 0$, i.e. $T^*(T(x^*)) = T^*(b)$, which has the equivalent matrix from $\overline{A}^T \cdot A \cdot x^* = \overline{A}^T \cdot b$. It is immediately that this last relation is the same with $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$. So from our theorem we reobtain the following well known result: if $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$ then $\|A \cdot x^* - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{C}^n$ (see [1], [2] or [3]).

Application 4. We consider the complex, finite, overdetermined linear system:

$$\begin{cases} a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n = b_0 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\ \vdots \end{cases}$$

where $a_{ij}, b_i \in \mathbb{C}$ for all $i \in \mathbb{N}$ and $j = \overline{1,n}$. Let $a_j = (a_{ij})_{i\in\mathbb{N}} \in l^2(\mathbb{C})$ and $b = (b_i)_{i\in\mathbb{N}} \in l^2(\mathbb{C})$, so $A = (a_1 a_2 \ldots a_n)$ is the matrix of the infinite linear system and b is the constant term. Then we obtain the following linear and continuous operator $T : \mathbb{C}^n \to l^2(\mathbb{C})$, $T(x) = A \cdot x$, for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$. So the adjoint operator T^* has the matrix $\overline{A}^T$, which is the transpose of the matrix A and taking the complex conjugate for all elements of A^T. Using our theorem, from the condition $(T(x^*) - b) \in \operatorname{Ker}(T^*)$ we obtain $T^*(T(x^*) - b) = 0$, i.e. $T^*(T(x^*)) = T^*(b)$, which has the equivalent matrix from $\overline{A}^T \cdot A \cdot x^* = \overline{A}^T \cdot b$. It is immediately that this last relation is the same with $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$. So from our theorem we obtain the following result: if $A^T \cdot \overline{A \cdot x^*} = A^T \cdot \overline{b}$ then $\|A \cdot x^* - b\| \leq \|A \cdot x - b\|$ for all $x \in \mathbb{C}^N$. (see [5]).

References

[1] Singiresu S. Rao, Applied Numerical Methods for Engineers and Scientists, Prentice Hall, Upper Saddle River, New Jersey, 2002.

[2] Anders C. Hansen, Infinite Dimensional Numerical Linear Algebra; Theory and Applications. `http://www.damtp.cam.ac.uk/user/na/people/Anders/Applied3.pdf`.

[3] Béla Finta, Numerical Analysis, Publishing House of the "Petru Maior" University, Tg. Mureş, Romania, 2004.

[4] Béla Finta, Overdetermined Infinite Linear Systems, ICNAAM 2009, Greece (submitted).

[5] Béla Finta, Complex Overdetermined Infinite Linear Systems, (to prepair).

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