

# Applications of Theorem Arens-Michael

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## Abstract

We characterize locally,  $m$ -convex, complete algebras as being projective limit of projective system of Banach algebras with the aid of Arens-Michael theorem.

## 1 Introduction

For a normed complex unitary algebra  $(A, \|\cdot\|)$  we define the set:  $D(A, \|\cdot\|; 1) = \{f \in A' | f(1) = 1 \text{ and } \|f\| = 1\}$ . For any  $a \in A$  we define numerical range of  $a$  the set

$$V(A, \|\cdot\|; a) = \{f(a) | f \in D(A, \|\cdot\|; 1)\},$$

and numerical radius the set:

$$v(A, \|\cdot\|; a) = \sup\{|\lambda| \mid \lambda \in V(A, \|\cdot\|; 1)\}.$$

The set  $D(A, \|\cdot\|; 1)$  is a convex subset, weak compact of  $A'$  and numerical range  $V(A, \|\cdot\|; a)$  is also a compact subset of  $\mathbf{C}$ , [2].

The properties and applications of numerical ranges on a normed algebra have been largely studied and the main results have been presented by F.F. Bonsall and J.Duncan [2]. The  $m$ -convex locally algebras have been thoroughly examined by E.A. Michael in [5].

We observe that for a given  $m$ -convex locally algebra  $A$ , with unital 1 there exists an increasingly family of submultiplicatively seminormes  $\{p_\alpha\}$  on  $A$  which generates the topology such that  $p_\alpha(1) = 1$  for all  $\alpha$ . Given this algebra we denote with  $P(A)$  the class of all these family of seminormes on  $A$

and with  $(A, \{p_\alpha\})$  the algebra  $A$  with the family  $\{p_\alpha\}$  fixed by seminormes  $\{p_\alpha\} \in P(A)$ .

Given  $(A, \{p_\alpha\})$  for each  $\alpha$  we denote with  $N_\alpha$  the null subspace of  $p_\alpha$ , through  $A_\alpha$  factor subspace  $A|_{N_\alpha}$  and with  $\|\cdot\|_\alpha$  we denote the norm on  $A_\alpha$ , defined by  $\|x + N_\alpha\|_\alpha = p_\alpha(x)$ . For each  $\alpha$ , we consider the linear canonical map  $x \mapsto x_\alpha \equiv x + N_\alpha$  from  $A$  to  $A_\alpha$ . We denote by  $1_\alpha$  the unital element in  $A_\alpha$  and it results that  $\|1_\alpha\|_\alpha = 1$  for all  $\alpha$ . Michael has obtained the significant result that  $A$  is isomorph with a subalgebra of the product of normed algebras  $(A_\alpha, \|\cdot\|_\alpha)$ .

We know that given the unitary algebra  $A$  for any  $a \in A$ , spectrum of  $a$  is defined as:

$\sigma(A; a) = \{\lambda \mid a - \lambda \cdot 1 \text{ is non-invertible}\}$ . We denote by  $\rho(x) := \sup_{\lambda \in \sigma(A, x)} |\lambda|$  the spectral radius of  $x$ .

We now that  $\rho(x) = \sup_\alpha \lim_{n \rightarrow \infty} (p_\alpha(x^n))^{1/n}$ .

## 2 Projective systems. Projective limits

Let be  $(A_i)_{i \in I}$  a family of algebras with  $I$  increasingly family, i.e for all  $i, j \in I \Rightarrow \exists k \in I$ , cu  $i \leq k, j \leq k$ . Let be a family  $\{f_{ij}\}_{i, j \in I}$  of morphism of algebras given by:

1.  $f_{ij} : A_j \rightarrow A_i$  for all  $i, j \in I$  with  $i \leq j$
2.  $f_{ii} = id_{A_i}, i \in I$
3.  $i, j, k \in I$ , cu  $i \leq j \leq k \Rightarrow f_{ik} = f_{ij} \circ f_{jk}$

The family  $\{(A_i, f_{ij})\}_{i, j \in I}$  is call *projective system algebras*. We consider cartesian product  $F = \prod_{i \in I} A_i$  and the following subset of  $F$ :

$$A = \{x \in (x_i)_{i \in I} \in F \mid x_i = f_{ij}(x_j), \text{ dac@}a \ i \leq j \in I\}.$$

**Theorem 1.**  $A$  is a subalgebra of  $F$ .

*Proof.* We show that  $A$  is a subalgebra of  $F$ . Let  $x, y \in A$ , it follows that  $x_i + y_i = f_{ij}(x_j + y_j) = f_{ij}(x_j) + f_{ij}(y_j)$ . Hence  $x + y \in A$ . Analogous we show that  $\alpha x \in A, \forall \alpha \in \mathbf{C}, \forall x \in A$ .

Let  $x, y \in A, xy = (x_i y_i)_{i \in I}, f_{ij}(x_j y_j) = f_{ij}(x_j) f_{ij}(y_j) = x_i y_i, \forall i, j \in I$  with  $i \leq j$ . Hence  $xy \in A$ . □

If each algebra  $A_i, i \in I$  have unital element,  $1_i$  and morphisms  $f_{ij}$ , with  $i \leq j, i, j \in I$  keeps unital, then unital of  $F, 1 = (1_i)_{i \in I}$  we find in subalgebra  $A$  of  $F$ .

The algebra  $A$  defined above is called projective limit algebra of projective system of algebras  $(A_i, f_{ij})$  and to denote through  $A = \varprojlim (A_i, f_{ij})$  or  $A = \varprojlim A_i$ . On the other hand doing a projective system of algebra to define a family  $(f_i)_{i \in I}$  of morphism of algebras through  $f_i = \pi_i|_A : A \rightarrow A_i$  for all  $i \in I$  i.e. considering restrictions to of  $A$  projects  $\pi_i$  of  $F$  on  $A_i$ .

The applications  $F_i$  can not to be surjectives.

From the definition of  $A$  and definition of applications  $f_i$ , it follows that  $f_i = f_{ij} \circ f_j$  for any  $i, j \in I$  with  $i \leq j$ .

$$f_{ij}(f_j(x)) = f_{ij}(x_j) = x_i = f_i(x).$$

**Definition 1.** Let be  $(A_i, f_{ij})$  a projective system of algebras, where  $A_i$  for any  $i \in I$  is a topological algebra and  $f_{ij}, i, j \in I, i \leq j$ , continuous morphism of algebras.

The system  $(A_i, f_{ij})$  is called a projective system of topological algebras.

The projective limits  $A$  is endowed with initial topology defined by family  $(f_i)_{i \in I}$ . This topological algebra is named projective limit of topological algebras.

*Proof.* A projective limit of locally m-convex algebras is an algebra. □

**Lemma 1.** Let  $A = \varprojlim (A_i, f_{ij})$  a projective limit of topological algebras.

Then the family  $\{f_i^{-1}(U_i) | U_i \in \mathcal{V}_i, i \in I\}$ , where  $\mathcal{V}_i$  represents a fundamentally system by neighborhoods of 0 from algebra  $A_i, i \in I$  is a fundamentally system by neighborhoods of 0 for  $A$ .

*Proof.* Let  $f_i = \pi_i|_A : \rightarrow A_i$ . We have that

$$\bigcap_{j=1}^n \pi_{i_j}^{-1}(U_{i_j}) \text{ implies } \bigcap_{j=1}^n f_{i_j}^{-1}(U_{i_j}), \quad (1)$$

where  $U_{i_j}$  belong of a fundamentally system of neighborhoods of 0, from  $A_{i_j}$ , is a fundamentally system of neighborhoods of 0 on  $\prod_{i \in I} A_i$ . Since  $I$  is a increasingly set  $i \in I$ , there exists  $i_j \leq i, j = 1, \dots, n$  such that  $f_{i_j} = f_{i_j i} \circ f_i$ . Hence

$$\bigcap_{j=1}^n f_{i_j}^{-1}(U_{i_j}) = \bigcap_{j=1}^n f_i^{-1} \left( f_{i_j i}^{-1}(U_{i_j}) \right) = \quad (2)$$

$$= f_i^{-1} \left( \bigcap_{j=1}^n f_{i_j i}^{-1}(U_{i_j}) \right) = f_i^{-1}(V_i), \quad (3)$$

whre  $V_i = \bigcap_{j=1}^n f_{ij}^{-1}(U_{ij})$  is a neighborhood in  $A_I$ . There exists  $U_i \in \mathcal{V}_i$ , such that  $U_i \subseteq V_i$  which shows assertions of enunciation.  $\square$

**Lemma 2.** Any projective limit of topological algebras  $A = \varprojlim A_i$  is a closed subalgebra of topological algebra cartesian product  $F = \prod_{i \in I} A_i$ . Particularly,  $A$  is complete if each  $A_i, i \in I$  from topological algebras is complete.

*Proof.* Let  $x \in \overline{A}$ . Then there exists  $(x^\alpha)_{\alpha \in J}$ ,  $x^\alpha \in A$ , such that  $x^\alpha \rightarrow x$  if and only if  $x^\alpha - x \rightarrow 0$  if and only if  $x_i^\alpha \rightarrow x_i$ , for all  $i \in I$ . From  $x^\alpha \in A$  it follows that

$$f_{ij}(x_j^\alpha) = x_i^\alpha, (\forall) i \leq j, i, j \in I. \quad (4)$$

From the continuity of  $f_{ij}$  it follows that

$$f_{ij}(x_j) = x_i, (\forall) i \leq j, i, j \in I.$$

Hence  $x \in A$ . If each  $A_i, i \in I$  is complete then the cartesian product  $A$  is a complete algebra and how  $A$  is a closed space it follows that is complete.  $\square$

**Lemma 3.** Let  $A = \varprojlim(A_i, f_{ij})$  a projective limit of topological algebra and  $B$  is a subalgebra of  $A$ . Then

$$\overline{B} = \bigcap_{i \in I} f_i^{-1} \left( \overline{f_i(B)} \right) = \varprojlim \overline{f_i(B)},$$

where  $f_i = \pi_i|_A : A \rightarrow A_i, (\forall) i \in I$ . Particularly, if  $B$  is closed, then

$$B = \varprojlim f_i(B) = \varprojlim \overline{f_i(B)}.$$

*Proof.* We observe that the family  $\left\{ \left( f_i(B), f_{ij}|_{f_j(B)} \right) \right\}_{i \in I}$  defines a projective system of topological algebras which it follows that from continuity of  $f_{ij}$  with  $i \leq j$  and from  $f_i = f_{ij} \circ f_j$  for  $i \leq j, i, j \in I$ .

On the other hand:

$$f_{ij} \left( \overline{f_j(B)} \right) \subseteq \overline{f_{ij}(f_j(B))} = \overline{f_i(B)}.$$

For  $i \leq j$  we obtain the family  $\left\{ \left( \overline{f_i(B)}, f_{ij}|_{\overline{f_j(B)}} \right) \right\}_{i \in I}$  defines also a projective system of topological algebras.

Immediately from definition it follows that

$$\varprojlim \overline{f_i(B)} = \bigcap_{i \in I} f_i^{-1} \left( \overline{f_i(B)} \right).$$

We show that  $\overline{B} = \bigcap_{i \in I} f_i^{-1}(\overline{f_i(B)})$ .

Let  $x \in \overline{B}$ . It follows that there exists a generalized sequence of  $B$ ,  $(x^\alpha)_{\alpha \in J}$ , such that  $x^\alpha \rightarrow x$ .

From the continuity of  $f_i$ , for all  $i \in I$ , it results  $f_i(x^\alpha) \rightarrow f_i(x)$ ,  $(\forall) x \in I \Rightarrow f_i(x) \in \overline{f_i(B)} \Rightarrow \Rightarrow x \in f_i^{-1}(\overline{f_i(B)})$ ,  $(\forall) i \in I$ , hence

$$x \in \bigcap_{i \in I} f_i^{-1}(\overline{f_i(B)}).$$

Conversely, let  $x \in \bigcap_{i \in I} f_i^{-1}(\overline{f_i(B)})$ . It follows that  $x \in f_i^{-1}(\overline{f_i(B)})$ ,

$(\forall) i \in I$ . It follows that  $f_i(x) \in \overline{f_i(B)}$ , for all  $i \in I$ , hence

$$x_i \in \overline{f_i(B)}, (\forall) i \in I,$$

it follows that for all  $U_i \in \mathcal{V}(x_i)$  we have  $U_i \cap f_i(B) \neq \emptyset$  it follows that  $f_i^{-1}(U_i) \cap B \neq \emptyset$ , hence  $x \in \overline{B}$  since  $f_i^{-1}(U_i)$  is a fundamental system by neighborhoods of  $x$ . On the other hand we have

$$B \subseteq \bigcap_{i \in I} f_i^{-1}(f_i(B)) \subseteq \bigcap_{i \in I} f_i^{-1}(\overline{f_i(B)}).$$

Then  $B \subseteq \varprojlim f_i(B) \subseteq \varprojlim \overline{f_i(B)} = \overline{B}$ , which prove last part of lemma.  $\square$

## 2.1 Arens-Michael Theorem

**Theorem 2.** (*Arens-Michael*)

Let be  $A$  a locally,  $m$ -convex algebra and  $\mathcal{V} = (U_i)_{i \in I}$  a fundamental system of neighborhoods of 0 equilibrates, convex, multiplicative and absorbent. We denote with  $\tilde{A}$  expanding of  $A$  and let be  $A_i$  (respective  $\tilde{A}_i$ ),  $i \in I$  normed algebras (Banach) suitable to fundamental system by neighborhoods  $\mathcal{V}$  defined above. Then

$$A \hookrightarrow \varprojlim A_i \hookrightarrow \varprojlim \tilde{A}_i = \tilde{A},$$

where  $\hookrightarrow$  is injective, topologic morphism of algebras. Particularly, for each locally,  $m$ -convex, complete algebra  $A$ , with fundamental system by neighborhoods  $\mathcal{V}$ , we obtain

$$A = \varprojlim A_i = \varprojlim \tilde{A}_i,$$

where " = " signifies topological isomorphism of algebras (surjective morphism).

*Proof.* Let be  $(p_i)_{i \in I}$  a increasingly family of seminorms associated of  $\mathcal{V}$ . Then  $A_i = A/N_i = \{x + N_i | x \in A\} = \{x_i\}, i \in I$  where  $N_i = p_i^{-1}(0)$ ,  $x_i = x + N_i, i \in I$ . Let be  $a \in A, x \in N_i$ . Then  $p_i(ax) \leq p_i(a) \cdot p_i(x)$ . Since  $p_i(x) = 0$  it follows that  $p_i(ax) = 0$ . Therefore,  $ax \in N_i$ . It follows that  $N_i$  is an ideal. Then  $A/N_i$  is an algebra.

We define  $\|x_i\|_i = p_i(x)$  which is obviously norm on  $A_i$ . Let  $f_{ij} : A_j \rightarrow A_i, \forall i, j \in I, j \geq i$ , given:

$$f_{ij}(x + N_j) = x + N_i.$$

Obviously,  $f_{ij}$  is a morphism of algebras. Let:

$$(1) f_{ij}((x_1 + N_j)(x_2 + N_j)) = f_{ik}(x_1x_2 + N_j) = x_1x_2 + N_i = (x_1 + N_i)(x_2 + N_i) = f_{ij}(x_1 + N_j)f_{ij}(x_2 + N_j)$$

We have that  $\|f_{ij}(x + N_j)\|_i = p_i(x) \leq p_j(x) = \|x + N_j\|_j$ ; it follows that  $f_{ij}$  continuous. We have:

$$(2) f_{ii}(x + N_i) = x + N_i.$$

Let  $i \leq j \leq k$ . Then

$$(3) f_{ik} = f_{ij} \circ f_{jk}(x + N_k) = f_{ij}(x + N_j) = x + N_i = f_{ik}(x + N_k),$$

hence  $f_{ik} = f_{ij} \circ f_{jk}$ , for all  $i \leq j \leq k, i, j, k \in I$ .

Therefore, from relations (1), (2), (3) the family  $\{(A_i, f_{ij})\}$  form a projective system of normed algebras.

Since  $(A_i)_{i \in I}$  (respective  $(\tilde{A}_i)_{i \in I}$ ) is a family of normed algebras (respective Banach) and form a projective system of normed algebras it follows that from above propositions that exists  $\varprojlim A_i$  (respectively  $\varprojlim \tilde{A}_i$ ) and is a locally  $m$ -convex algebra.

Considering locally  $m$ -convex algebras  $\varprojlim A_i$  and  $\varprojlim \tilde{A}_i$ , we define the following application:

$$\varphi : A \rightarrow \varprojlim A_i$$

through

$$\varphi(x) = (\varphi_i(x))_{i \in I} = (x_i)_{i \in I}$$

where  $\varphi_i(x) = x + N_i = x + \text{Ker}(p_i) = x_i, i \in I$ .

To show that  $\varphi$  is well defined to observe that  $\varphi_i = f_{ij} \circ \varphi_j$

Indeed,  $f_{ij}(\varphi_j(x)) = \varphi_i(x)$ . It follows that  $\varphi(x) \in \varprojlim A_i$ . It is obviously that  $\varphi$  is a morphism of algebras. From  $\varphi(u) = 0$  it follows that  $(\varphi_i(x))_{i \in I} = 0$  hence  $\varphi_i(x) = 0, i \in I$ . Hence  $p_i(x) = 0, i \in I$ . Therefore,  $x = 0$  ( $A$  algebra separate) and so  $\varphi$  is a injective morphism, hence isomorphic of algebras.

We prove now that  $\varphi$  is topological isomorphism.

We have  $f_i \circ \varphi_i = \varphi_i$ , for all  $i \in I$ .

Indeed,  $(f_i \circ \varphi)(x) = f_i(\varphi(x)) = f_i(\varphi_i(x))_{i \in I} = \varphi_i(x)$ . Since  $f_i$  and  $\varphi_i$  are continue functions  $i \in I$  it follows that  $\varphi$  is continuous.

The inverse functions  $\varphi^{-1} : \lim_{\leftarrow} A_i \rightarrow A$  is also continuous. Indeed if  $U_i \in \mathcal{V}$  then

$$V = \left( \prod_{j \in I} V_j \right) \cap \varphi(A),$$

with  $U_i = \varphi_i \left( \frac{1}{2} U_i \right)$  and  $V_j = A_j$ , for any  $j \in V, j \neq i$  is a neighborhoods of 0 in  $\varphi(A)$  with properties that  $V \subseteq \varphi(U_i)$ , which means  $\varphi^{-1}(V) \subseteq U_i$  and hence  $\varphi^{-1}$  is continuous.

We verify that  $V \subseteq \varphi(U_i)$ . Let  $y \in V \Rightarrow y \in \prod_{j \in J} V_j$  and  $y \in \varphi(A)$ ,

it follows that there exists  $x \in A$  with  $y = \varphi(x)$  and  $\varphi_i(x) = \varphi_i(\frac{1}{2}z)$ , with  $z \in U_i$ .

Hence  $p_i(x) = p_i \left( \frac{1}{2}z \right) = \frac{1}{2}p_i(z) \leq \frac{1}{2} < 1 \Rightarrow$   
 $\Rightarrow x \in U_i \Rightarrow y = \varphi(x)$  with  $x \in U_i \Rightarrow y = \varphi(x) \in \varphi(U_i) \Rightarrow$   
 $\Rightarrow V \subseteq \varphi(U_i) \Rightarrow \varphi^{-1}(V) \subseteq U_i$ .

Therefore,  $\varphi$  is surjective, topologic morphism of algebras of  $A$  to into  $\lim_{\leftarrow} A_i$ .

We have canonical injections  $\theta_i : A_i \rightarrow \tilde{A}_i, i \in I$  which are algebraical and topological isomorphisms and commutes with the maps  $f_{ij}$  and with their extensions  $\tilde{f}_{ij} : \tilde{a} \rightarrow \tilde{A}_i$ , with  $i \leq j$ .

We obtain:

$$\theta : \lim_{\leftarrow} A_i \rightarrow \lim_{\leftarrow} \tilde{A}_i$$

defined through  $\theta(x) = (\theta_i(x_i)), i \in I$ .

$$\tilde{f}_{ij}(\theta_j(x_j)) = \tilde{f}_{ij}(x_j) = f_{ij}(x_j) = x_i = \theta_i(x_i)$$

Isomorphism  $\theta$  is topologic too, which it follows immediately from the definitions of algebraical topologies.

On the other hand, since

$$f_i \circ \varphi = \varphi_i, (\forall) i \in I, \quad \varphi_i : A \rightarrow A_i = A / \text{Ker}_{\varphi_i}$$

we have:

$$f_i(\varphi(A)) = \varphi_i(A) = A_i, i \in I$$

and then from above lema and from conclusions for  $\varphi$  we obtain

$$A \subseteq \bar{A} = \overline{\varphi(A)} = \lim_{\leftarrow} \overline{f_i(\varphi(A))} = \lim_{\leftarrow} A_i = \lim_{\leftarrow} \tilde{A}_i.$$

Since the last space is complete it follows that from above lemma  $\tilde{A} = \bar{A} = \lim_{\leftarrow} \tilde{A}_i = \lim_{\leftarrow} \tilde{A}$  where equality represents isomorphisms of studied algebras above.

Therefore from  $\theta$  is a topological isomorphism the proof is end.  $\square$

## 2.2 Applications of Arens-Michael theorem

**Theorem 3.** Let be  $A$  a locally,  $m$ -convex and complete algebra and  $A = \varprojlim A_i$ .

- 1) The algebra  $A$  has unital element if and only if  $\tilde{A}_i$  has unital element for all  $i \in I$ .
- 2) An element  $x \in A$  is invertible if and only if  $\varphi_i(x)$  is invertible in  $\tilde{A}_i$  for any  $i \in I$ .

*Proof.* We suppose  $1 = (1_i) \in \prod_{i \in I} \tilde{A}_i$ , with  $1_i$  unital element in  $\tilde{A}_i$  for  $i \in I$ .

Since  $A_i = \varphi_i(A)$ , if  $x_i = \varphi_i(x) \in A_i$ , then

$$\begin{aligned} x_i \tilde{f}_{ij}(1_j) &= \varphi_i(x) \tilde{f}_{ij}(1_j) = \tilde{f}_{ij}(\varphi_j(x)) \tilde{f}_{ij}(1_j) = \tilde{f}_{ij}(\varphi_j(x)1_j) = \\ &= \tilde{f}_{ij}(\varphi_j(x)) = \varphi_i(x) = x_i \end{aligned}$$

for any  $i \leq j$  and similarly for left multiplication with  $\tilde{f}_{ij}(1_j)$ , hence  $\tilde{f}_{ij}(1_j)$  is a unital for  $A_i$ , hence also for  $\tilde{A}_i = \overline{A}_i$ . Then it follows that  $\tilde{f}_{ij}(1_j) = 1_i$  for any  $i \leq j$  from  $I$ , hence  $1 = (1_i)_{i \in I} \in \varprojlim \tilde{A}_i = A$ .

Verify that  $1$  is unital element of  $A$ .

We prove (2). If  $x = (x_i)_{i \in I} \in A = \varprojlim A_i$ , how  $x_i$  is invertible of  $\tilde{A}_i$  for any  $i \in I$ , there exist  $y = (y_i)_{i \in I} \in \prod_{i \in I} \tilde{A}_i$  such that:

$$x_i \cdot y_i = y_i \cdot x_i = 1_i,$$

where from (1) we know that  $(1_i)_{i \in I} = 1$  is a unital element of  $A$ .

Now for  $i \leq j$  from  $I$ , we obtain:

$$\begin{aligned} x_i \cdot \tilde{f}_{ij}(y_j) &= \tilde{f}_{ij}(x_j) \cdot \tilde{f}_{ij}(y_j) = \tilde{f}_{ij}(x_j \cdot y_j) = \\ &= \tilde{f}_{ij}(1_j) = 1_i = \tilde{f}_{ij}(y_j) \cdot x_i, \end{aligned}$$

which means that  $\tilde{f}_{ij}(y_j)$  is inverse of  $x_i$  in  $\tilde{a}_i$  and then we deduce  $\tilde{f}_{ij}(y_j) = y_i$ , for  $i \leq j$  from  $I$ .

Hence  $y = (y_i)_{i \in I} \in \varprojlim A_i$ . From above we deduce that  $y$  is inverse of  $x$  in  $A$ . □

**Corollary 1.** Let be  $A$  a locally,  $m$ -convex and complete algebra and  $x \in A$ . Then:

$$\sigma(A, x) = \bigcup_{i \in I} \sigma(A, x_i),$$

$$\rho(x) = \sup_{i \in I} \rho(x_i) = \sup_{i \in I} \lim_{n \rightarrow \infty} (p_i(x^n))^{\frac{1}{n}}.$$

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