Property of Regular Excessive Measures

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Abstract

We show that a measure $\xi \in \mathrm{Exc}_{\mathcal{U}}$ will be regular if and only if exists $\mu \in \mathcal{M}_+(E)$ such that $\xi = \mu \circ U$, where μ is a σ-finite measure on E which does not charge any $\mathcal{B}$-measurable semipolar set.

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Introduction

We consider a proper sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin measurable space $(E, \mathcal{B})$ such that the set $\mathcal{E}_{\mathcal{U}}$ of all $\mathcal{B}$-measurable, $\mathcal{U}$-excessive functions on E which are $\mathcal{U}$-a.e finite is min-stable, contains the positive constant functions and generates the σ-algebra $\mathcal{B}$.

We suppose that the set E is semisaturated (see [3]) with respect to $\mathcal{U}$ i.e. any $\mathcal{U}$-excessive measure dominated by a potential is also potential. We denote by $\mathrm{Exc}_{\mathcal{U}}$ the set of all $\mathcal{U}$-excessive measures on E (see [11], [12]). The specific order $\preceq$ on $\mathrm{Exc}_{\mathcal{U}}$ is defined as follows: if $\xi, \xi' \in \mathrm{Exc}_{\mathcal{U}}$ then $\xi \preceq \xi'$ if and only if there exists $\eta \in \mathrm{Exc}_{\mathcal{U}}$ such that $\xi + \eta = \xi'$.

For every σ-finite measure μ on $(E, \mathcal{B})$ such that exists $\xi \in \mathrm{Exc}_{\mathcal{U}}$ with $\mu \leq \xi$, we put $R(\mu) = \wedge\{\xi \in \mathrm{Exc}_{\mathcal{U}} \mid \mu \leq \xi\}$.

1 Semipolar sets

Throughout the paper $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be a proper sub-Markovian resolvent on $(E, \mathcal{B})$ as in Introduction. If $A \subset E$ and $s \in \mathcal{E}_{\mathcal{U}}$ then réduite of s on A is the function $R^A s$ on E defined by $R^A s := \inf\{t \in \mathcal{E}_{\mathcal{U}} \mid t \geq s \text{ on } A\}$. If $A \in \mathcal{B}$ and $s \in \mathcal{E}_{\mathcal{U}}$ then $R^A s$ is universally measurable (see [5]) and we denote by $B^A s$ its $\mathcal{U}$-excessive regularization. (The fine toplogy is the topology on E generated by $\mathcal{E}_{\mathcal{U}}$). The function $B^A s$ is called the baleyage of s on A.

For every $A \in \mathcal{B}^u$ we denote by A^* the set given by

$$A^* = \{x \in A \mid \liminf_n nU_n(1_A)(x) = 1\},$$

where 1_A is the characteristic function of A. Clearly $A^* \in B^u$ and $A^* \in \mathcal{B}$ provided that $A \in \mathcal{B}$.

Theorem 1. *For every $A \in \mathcal{B}^u$ the following assertions hold:*

1. *If A is finely open then $A = A^*$.*

2. *The set $A \setminus A^*$ is $\mathcal{U}$-negligible and for each $s \in \mathcal{E}_{\mathcal{U}}$ the function $R^{A^*}s$ is $\mathcal{U}$-excessive and there exists a sequence $(f_n)_n$ in $bp\mathcal{B}^u$ such that*

$$Uf_n \uparrow R^{A^*}s \quad \textit{and} \quad f_n = 0 \quad \textit{on} \quad E \setminus A^* \tag{1}$$

3. *If $A \in \mathcal{B}$ and $s \in \mathcal{E}_{\mathcal{U}}$ then $R^{A^*}s \in \mathcal{E}_{\mathcal{U}}$ and there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ such that (1) holds.*

For the proof see Theorem 1.3.8 [6].

It is known that if $A \in \mathcal{B}$ then one has $B^A s = R^A s$ on $E \setminus A$ (see [5]]).

If $\xi \in \mathrm{Exc}_{\mathcal{U}}$ a subset A of E is called ξ-*polar* provided that there exists a $\mathcal{U}$-excessive function s on E such that $s = +\infty$ on A and $s < \infty$ ξ- a.e. If μ is a σ-finite measure on E such that $\mu \circ U \in \mathrm{Exc}_{\mathcal{U}}$ then we say μ-*polar* instead of $\mu \circ U$ polar. A set A is called *polar* if it is ξ-polar for every $\xi \in \mathrm{Exc}_{\mathcal{U}}$.

A subset M of E is called nearly $\mathcal{B}$-measurable provided that for every finite measure μ on $(E, \mathcal{B})$ there exists a $\mathcal{B}$-measurable set $M_0 \subset M$ such that the set $M \setminus M_0$ is μ-polar and μ-negligible.

We denote by $\mathcal{B}^n$ the family of all nearly $\mathcal{B}$-measurable subsets of E (see [6]). Obviously $\mathcal{B}^n$ is a σ-algebra on E and $\mathcal{B} \subset \mathcal{B}^n \subset \mathcal{B}^u$ (the set of all universally $\mathcal{B}$-measurable subsets of E).

Recall that a set $A \in \mathcal{B}$ is *thin* at a point $x \in E$ if there exists $s \in \mathcal{E}_{\mathcal{U}}$ such that $B^A s(x) < s(x)$. A subset M of E is called thin at x if there exists $A \in \mathcal{B}$ such that $M \subset A$, which is thin at x. The set M is called *totally thin* if it is thin at any point of E. For any subset M of E the set $b(M) := \{x \in E \mid M \text{ is not thin at } x\}$ is usually called the *base* of M. It is a fine closed set and $b(M) = b(\overline{M}^f) \subset \overline{M}^f$, where $\overline{M}^f$ denotes the fine closure of M. If M is nearly $\mathcal{B}$-measurable and $p := Uf_0$ is bounded with f_0 $\mathcal{B}$-measurable, $0 < f_0 \leq 1$ then we have

M is thin at $x \Leftrightarrow B^M p(x) < p(x)$

$b(M) = [B^M p = p]$.

A subset M of E is called *basic* (respective *subbasic*) if $b(M) = M (resp.\, M \subset b(M))$. If M is subbasic then $\overline{M}^f$ is basic.

A subset of E is termed *semipolar* if it is a countable union of totally thin sets.

Remark. Since $R^{A^*} s \in \mathcal{E}_{\mathcal{U}}$ for any $A \in \mathcal{B}^u$ (see [5]) it follows that $R^{A^*} s = \mathcal{B}^{A^*} s$ for all $s \in \mathcal{E}_{\mathcal{U}}$. Hence $B^{A^*} U f_0 = U f_0$ on A^*, where $0 < f_0 \leq 1$, f_0 $\mathcal{B}$-measurable such that $U f_0$ is bounded. Therefore A^* is a subbasic set.

2 The regular excessive measures

We recall that an element $\xi \in \mathrm{Exc}_{\mathcal{U}}$ is called *regular* if for any increasing sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathrm{Exc}_{\mathcal{U}}$ such that $\vee_{n\in\mathbb{N}} \xi_n = \xi$ we have $\wedge_{n\in\mathbb{N}} R(\xi - \xi_n) = 0$ (see [3]).

Note (see [5], [6]) that a $\mathcal{U}$-excessive measure $\xi = \mu \circ U$ is regular iff μ is a σ-finite measure on E which does not charge any $\mathcal{B}$-measurable semipolar set.

For the proof see Theorem 3.4.5 [6].

Theorem 2. *Let E be a semisaturated set with respect to $\mathcal{U}$. Then a measure $\xi \in \mathrm{Exc}_{\mathcal{U}}$ is regular if and only if exists $\mu \in \mathcal{M}_+(E)$ such that $\xi = \mu \circ U$, where μ is a σ-finite measure on E which does not charge any $\mathcal{B}$-measurable semipolar set.*

Proof. Let $(\mu_n)_n$ a sequence of $\mathcal{M}_+(E)$ such that $\mu_n \circ U \uparrow \xi$. Because ξ is regular we get $R(\xi - \mu_n \circ U) \downarrow 0$. Putting $\xi_n = R(\xi - \mu_n \circ U)$ we get $\xi_n \preceq \xi$ and excessive measure $\eta_n := \xi - \xi_n$ is such that $\eta_n \leq \mu_n \circ U$ i.e. there exists $\nu_n \in \mathcal{M}_+(E)$ with $\eta_n = \nu_n \circ U$ and $\eta_n \preceq \xi$. From $\xi_n \downarrow 0$ we have that $\eta_n \uparrow \xi$. We denote by $\xi' = \curlyvee_n \eta_n$. Because $\eta_n + \xi_n = \xi$ we get $\curlyvee_{k=1}^n \eta_k + \curlywedge_{k=1}^n \xi_k = \xi$. But from $\xi_n \downarrow 0$ we have that $\curlyvee_n \eta_n = \xi$. Therefore $\xi' = \xi$. Putting $\curlyvee_{k\leq n} \eta_k = \eta'_n$ it results that there exists $\nu'_n \in \mathcal{M}_+(E)$ such that $\eta'_n = \nu'_n \circ U$. Because $\curlyvee_n \eta_n = \curlyvee_n \eta'_n$ and putting $\mu = \bigvee_n \nu'_n$ we get $\xi = \mu \circ U$. Therefore ξ is regular if and only if $\mu \circ U$ is regular i.e. iff μ does not charge any $\mathcal{B}$-measurable semipolar set. $\square$

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