

# Property of Regular Excessive Measures

Diana Mărginean Petrovai

Proceedings of the European Integration between Tradition and  
Modernity

"Petru Maior" University of Târgu-Mureș

Oktober, 22-23, 2009

## Abstract

We show that a measure  $\xi \in \text{Exc}_{\mathcal{U}}$  will be regular if and only if exists  $\mu \in \mathcal{M}_+(E)$  such that  $\xi = \mu \circ U$ , where  $\mu$  is a  $\sigma$ -finite measure on  $E$  which does not charge any  $\mathcal{B}$ -measurable semipolar set.

*AMS 2000 Subject Classification:* 31D05, 60J45

*Key words:* excessive measures, excessive functions, semipolar set

## Introduction

We consider a proper sub-Markovian resolvent of kernels  $\mathcal{U} = (U_\alpha)_{\alpha>0}$  on a Lusin measurable space  $(E, \mathcal{B})$  such that the set  $\mathcal{E}_{\mathcal{U}}$  of all  $\mathcal{B}$ -measurable,  $\mathcal{U}$ -excessive functions on  $E$  which are  $\mathcal{U}$ -a.e finite is min-stable, contains the positive constant functions and generates the  $\sigma$ -algebra  $\mathcal{B}$ .

We suppose that the set  $E$  is semisaturated (see [3]) with respect to  $\mathcal{U}$  i.e. any  $\mathcal{U}$ -excessive measure dominated by a potential is also potential. We denote by  $\text{Exc}_{\mathcal{U}}$  the set of all  $\mathcal{U}$ -excessive measures on  $E$  (see [11], [12]). The specific order  $\preceq$  on  $\text{Exc}_{\mathcal{U}}$  is defined as follows: if  $\xi, \xi' \in \text{Exc}_{\mathcal{U}}$  then  $\xi \preceq \xi'$  if and only if there exists  $\eta \in \text{Exc}_{\mathcal{U}}$  such that  $\xi + \eta = \xi'$ .

For every  $\sigma$ -finite measure  $\mu$  on  $(E, \mathcal{B})$  such that exists  $\xi \in \text{Exc}_{\mathcal{U}}$  with  $\mu \leq \xi$ , we put  $R(\mu) = \wedge \{\xi \in \text{Exc}_{\mathcal{U}} \mid \mu \leq \xi\}$ .

# 1 Semipolar sets

Throughout the paper  $\mathcal{U} = (U_\alpha)_{\alpha>0}$  will be a proper sub-Markovian resolvent on  $(E, \mathcal{B})$  as in Introduction. If  $A \subset E$  and  $s \in \mathcal{E}_{\mathcal{U}}$  then réduite of  $s$  on  $A$  is the function  $R^A s$  on  $E$  defined by  $R^A s := \inf\{t \in \mathcal{E}_{\mathcal{U}} \mid t \geq s \text{ on } A\}$ . If  $A \in \mathcal{B}$  and  $s \in \mathcal{E}_{\mathcal{U}}$  then  $R^A s$  is universally measurable (see [5]) and we denote by  $B^A s$  its  $\mathcal{U}$ -excessive regularization. (The fine topology is the topology on  $E$  generated by  $\mathcal{E}_{\mathcal{U}}$ ). The function  $B^A s$  is called the baleage of  $s$  on  $A$ .

For every  $A \in \mathcal{B}^u$  we denote by  $A^*$  the set given by

$$A^* = \{x \in A \mid \liminf_n nU_n(1_A)(x) = 1\},$$

where  $1_A$  is the characteristic function of  $A$ . Clearly  $A^* \in \mathcal{B}^u$  and  $A^* \in \mathcal{B}$  provided that  $A \in \mathcal{B}$ .

**Theorem 1.** *For every  $A \in \mathcal{B}^u$  the following assertions hold:*

1. *If  $A$  is finely open then  $A = A^*$ .*
2. *The set  $A \setminus A^*$  is  $\mathcal{U}$ -negligible and for each  $s \in \mathcal{E}_{\mathcal{U}}$  the function  $R^{A^*} s$  is  $\mathcal{U}$ -excessive and there exists a sequence  $(f_n)_n$  in  $bp\mathcal{B}^u$  such that*

$$Uf_n \uparrow R^{A^*} s \quad \text{and} \quad f_n = 0 \quad \text{on} \quad E \setminus A^* \tag{1}$$

3. *If  $A \in \mathcal{B}$  and  $s \in \mathcal{E}_{\mathcal{U}}$  then  $R^{A^*} s \in \mathcal{E}_{\mathcal{U}}$  and there exists a sequence  $(f_n)_n$  in  $bp\mathcal{B}$  such that (1) holds.*

For the proof see Theorem 1.3.8 [6].

It is known that if  $A \in \mathcal{B}$  then one has  $B^A s = R^A s$  on  $E \setminus A$  (see [5]).

If  $\xi \in \text{Exc}_{\mathcal{U}}$  a subset  $A$  of  $E$  is called  $\xi$ -polar provided that there exists a  $\mathcal{U}$ -excessive function  $s$  on  $E$  such that  $s = +\infty$  on  $A$  and  $s < \infty$   $\xi$ - a.e. If  $\mu$  is a  $\sigma$ -finite measure on  $E$  such that  $\mu \circ U \in \text{Exc}_{\mathcal{U}}$  then we say  $\mu$ -polar instead of  $\mu \circ U$  polar. A set  $A$  is called polar if it is  $\xi$ -polar for every  $\xi \in \text{Exc}_{\mathcal{U}}$ .

A subset  $M$  of  $E$  is called nearly  $\mathcal{B}$ -measurable provided that for every finite measure  $\mu$  on  $(E, \mathcal{B})$  there exists a  $\mathcal{B}$ -measurable set  $M_0 \subset M$  such that the set  $M \setminus M_0$  is  $\mu$ -polar and  $\mu$ -negligible.

We denote by  $\mathcal{B}^n$  the family of all nearly  $\mathcal{B}$ -measurable subsets of  $E$  (see [6]). Obviously  $\mathcal{B}^n$  is a  $\sigma$ -algebra on  $E$  and  $\mathcal{B} \subset \mathcal{B}^n \subset \mathcal{B}^u$  (the set of all universally  $\mathcal{B}$ -measurable subsets of  $E$ ).

Recall that a set  $A \in \mathcal{B}$  is *thin* at a point  $x \in E$  if there exists  $s \in \mathcal{E}_{\mathcal{U}}$  such that  $B^A s(x) < s(x)$ . A subset  $M$  of  $E$  is called thin at  $x$  if there exists  $A \in \mathcal{B}$  such that  $M \subset A$ , which is thin at  $x$ . The set  $M$  is called *totally thin* if it is thin at any point of  $E$ . For any subset  $M$  of  $E$  the set  $b(M) := \{x \in E \mid M \text{ is not thin at } x\}$  is usually called the *base* of  $M$ . It is a fine closed set and  $b(M) = b(\overline{M}^f) \subset \overline{M}^f$ , where  $\overline{M}^f$  denotes the fine closure of  $M$ . If  $M$  is nearly  $\mathcal{B}$ -measurable and  $p := Uf_0$  is bounded with  $f_0$   $\mathcal{B}$ -measurable,  $0 < f_0 \leq 1$  then we have

$$M \text{ is thin at } x \Leftrightarrow B^M p(x) < p(x)$$

$$b(M) = [B^M p = p].$$

A subset  $M$  of  $E$  is called *basic* (respective *subbasic*) if  $b(M) = M$  (resp.  $M \subset b(M)$ ). If  $M$  is subbasic then  $\overline{M}^f$  is basic.

A subset of  $E$  is termed *semipolar* if it is a countable union of totally thin sets.

*Remark.* Since  $R^{A^*} s \in \mathcal{E}_{\mathcal{U}}$  for any  $A \in \mathcal{B}^u$  (see [5]) it follows that  $R^{A^*} s = \mathcal{B}^{A^*} s$  for all  $s \in \mathcal{E}_{\mathcal{U}}$ . Hence  $B^{A^*} Uf_0 = Uf_0$  on  $A^*$ , where  $0 < f_0 \leq 1$ ,  $f_0$   $\mathcal{B}$ -measurable such that  $Uf_0$  is bounded. Therefore  $A^*$  is a subbasic set.

## 2 The regular excessive measures

We recall that an element  $\xi \in \text{Exc}_{\mathcal{U}}$  is called *regular* if for any increasing sequence  $(\xi_n)_{n \in \mathbb{N}} \subset \text{Exc}_{\mathcal{U}}$  such that  $\bigvee_{n \in \mathbb{N}} \xi_n = \xi$  we have  $\bigwedge_{n \in \mathbb{N}} R(\xi - \xi_n) = 0$  (see [3]).

Note (see [5], [6]) that a  $\mathcal{U}$ -excessive measure  $\xi = \mu \circ U$  is regular iff  $\mu$  is a  $\sigma$ -finite measure on  $E$  which does not charge any  $\mathcal{B}$ -measurable semipolar set.

For the proof see Theorem 3.4.5 [6].

**Theorem 2.** *Let  $E$  be a semisaturated set with respect to  $\mathcal{U}$ . Then a measure  $\xi \in \text{Exc}_{\mathcal{U}}$  is regular if and only if exists  $\mu \in \mathcal{M}_+(E)$  such that  $\xi = \mu \circ U$ , where  $\mu$  is a  $\sigma$ -finite measure on  $E$  which does not charge any  $\mathcal{B}$ -measurable semipolar set.*

*Proof.* Let  $(\mu_n)_n$  a sequence of  $\mathcal{M}_+(E)$  such that  $\mu_n \circ U \uparrow \xi$ . Because  $\xi$  is regular we get  $R(\xi - \mu_n \circ U) \downarrow 0$ . Putting  $\xi_n = R(\xi - \mu_n \circ U)$  we get  $\xi_n \preceq \xi$  and excessive measure  $\eta_n := \xi - \xi_n$  is such that  $\eta_n \leq \mu_n \circ U$  i.e. there exists  $\nu_n \in \mathcal{M}_+(E)$  with  $\eta_n = \nu_n \circ U$  and  $\eta_n \preceq \xi$ . From  $\xi_n \downarrow 0$  we have that  $\eta_n \uparrow \xi$ . We denote by  $\xi' = \bigvee_n \eta_n$ . Because  $\eta_n + \xi_n = \xi$  we get  $\bigvee_{k=1}^n \eta_k + \bigwedge_{k=1}^n \xi_k = \xi$ . But from  $\xi_n \downarrow 0$  we have that  $\bigvee_n \eta_n = \xi$ . Therefore  $\xi' = \xi$ . Putting  $\bigvee_{k \leq n} \eta_k = \eta'_n$  it results that there exists  $\nu'_n \in \mathcal{M}_+(E)$  such that  $\eta'_n = \nu'_n \circ U$ . Because  $\bigvee_n \eta_n = \bigvee_n \eta'_n$  and putting  $\mu = \bigvee_n \nu'_n$  we get  $\xi = \mu \circ U$ . Therefore  $\xi$  is regular if and only if  $\mu \circ U$  is regular i.e. iff  $\mu$  does not charge any  $\mathcal{B}$ -measurable semipolar set.  $\square$

## References

- [1] L. Beznea and N. Boboc, *Absorbent, parabolic, elliptic and quasi-elliptic balayages in potential theory*, Rev. Roumaine Math. Pures Appl. **38** (1993), 197-234.
- [2] L. Beznea and N. Boboc, *Duality and biduality for excessive measures*, Rev. Roumaine Math. Pures Appl. **39** (1994), 419-438.
- [3] L. Beznea and N. Boboc, *On the integral representation for excessive measures*, Rev. Roumaine Math. Pures Appl. **40** (1995), 725-734.
- [4] L. Beznea and N. Boboc, *Balayages on Excessive Measures, their Representation and the Quasi-Lindelöf Property*, Kluwer Academic Publishers, 1997.
- [5] L. Beznea and N. Boboc, *Excessive functions and excessive measures: Hunt's theorem on balayages, quasi-continuity*, in Proc. Workshop on Classical and Modern Potential Theory and Applications, NATO ASI Series C 430, pp. 77-92, Kluwer, 1994.
- [6] N. Boboc and L. Beznea, *Potential Theory and Right Processes*, Dordrecht, Kluwer, 2004.
- [7] N. Boboc and Gh. Bucur, A. Cornea, *Order and Convexity in Potential Theory:  $H$ -cones*, Springer-Verlag, Berlin - Heidelberg - New York, 1981.
- [8] N. Boboc and Gh. Bucur, *Măsură și capacitate, Editura științifică și enciclopedică*, București, 1985.
- [9] N. Boboc and Gh. Bucur, *Excessive and supermedian functions with respect to subordinated resolvents of kernels*, Rev. Roumaine Math. Pures Appl., **39** (1994), 875-878.
- [10] A. Cornea and G. Licea, *Order and Potential Resolvent Families of Kernels*, Springer-Verlag, Berlin - Heidelberg - New York, 1975.
- [11] C. Dellachaire and P.A. Meyer, *Probabilités et Potential*, Ch I-IV, IX-XI, XII-XVI, Hermann, 1975, 1983, 1987.
- [12] R.K. Gettoor, *Excessive Measures*, La Jolla, California, 1989.

- [13] G. Mokobodzki, *Pseudo-quotient de deux mesures, application à la dualité*, in Séminaire de probabilités VII, Lecture Notes in Math, 321, 318-321, Springer, 1973.
- [14] D. Marginean Petrovai, *New Properties of Excessive Measures*, *Mathematical Reports*, nr. 4, 2008

"Petru Maior" University of Tg. Mureş  
N. Iorga street nr. 1, 540088, Romania  
*dianapetrovai@yahoo.com*