

# Direct Decompositions of Quasigroups and Homotopies

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## Abstract

In this paper we investigate direct decompositions in the category **QGR** whose objects are  $n$ -quasigroups and morphisms are quasigroup homotopies.

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## 1 Introduction

In the theory of  $n$ -quasigroups ( $n \geq 2$ ) the role played by homotopies is as important as that played by homomorphisms. But, in many applications of  $n$ -quasigroups, isotopies and homotopies are more important than isomorphisms and homomorphisms. So, the study of homotopic properties of algebraic constructions become important.

Direct products give a means of creating  $n$ -quasigroups of huge order (applications in cryptography) than what we start with. A direct product of a family of  $n$ -quasigroups is completely determined by its factors.

The aim of the present paper is to present direct decomposition of  $n$ -quasigroups in the category **QGR** whose morphisms are quasigroup homotopies.

The second section records absolute and weak permutability of equivalence relations. Quasigroup homotopies kernels are presented in section 3. Section 4 examines direct decompositions.

To simplify the notation, we will omit the prefix  $n$  in  $n$ -quasigroup.

## 2 Permutability of equivalence relations

We recall two generalizations of the permutability of equivalence relations.

Let  $\mathcal{S} = \{q_j \mid j \in J\}$  be a family of equivalence relations on a set  $A$ .

$\mathcal{S}$  is called **absolutely permutable** [4] if it satisfies the following condition: for any family  $\{a_j \mid j \in J\}$  if  $a_j \equiv a_k(\vee q_j)$  for all  $j, k \in J$  there exists  $a \in A$  such that  $a \equiv a_j(q_j)$  for all  $j \in J$ .

$\mathcal{S}$  is called **weakly permutable** [1] if it satisfies the following condition: for any family  $\{a_j \mid j \in J\}$  if  $a_j \equiv a_k(q_j \vee q_k)$  for all  $j, k \in J$  there exists  $a \in A$  such that  $a \equiv a_j(q_j)$  for all  $j \in J$ .

The concept of weak permutability is weaker than that of absolute permutability. Some useful results are:

**Theorem 1.** *The following are equivalent:*

- (i)  $\mathcal{S}$  is absolutely permutable;
- (ii)  $\mathcal{S}$  is weakly permutable and for any  $q_j, q_k \in \mathcal{S}$ , if  $q_j \neq q_k$ , then  $q_j \vee q_k = \vee\{q_j \mid j \in J\}$ .

**Theorem 2.** *If  $\mathcal{S}$  is absolutely permutable then  $q_j \circ \bar{q}_j = \vee\{q_j \mid j \in J\}$ , where  $\bar{q}_j = \wedge\{q_k \mid k \in J, k \neq j\}$ , for all  $j \in J$ . If  $J$  is finite the converse is true.*

We will simplify several proofs in section 4 using the following result.

**Theorem 3.** *Let  $A$  and  $J$  be two sets. There exists a bijective map  $f : A \rightarrow \prod B_j$  the cartesian product of the family  $\{B_j \mid j \in J\}$  of sets if and only if there exists a family  $\mathcal{S} = \{q_j \mid j \in J\}$  of equivalence relations on  $A$  such that:*

- (i)  $\wedge q_j = \Delta_A$ ;
- (ii)  $\vee q_j = A^2$ ;
- (iii)  $\mathcal{S}$  is absolutely permutable.

*Proof.* Suppose  $f : A \rightarrow \prod B_j$ . Let be  $\mathcal{S} = \{q_j \mid j \in J\}$  where  $q_j = \ker(p_j f)$ ,  $p_j$  being the  $j$ -th projection. We have  $\Delta_A = \ker(f) = \wedge \ker(p_j f) = \wedge q_j$ . Let  $a, a' \in A$ . Choose  $j, k \in J$ , and consider an element  $b \in \prod B_j$  such that  $p_j(b) = p_j f(a)$  and  $p_k(b) = p_k f(a')$ . The map  $f$  being surjective there exists  $a^* \in A$  such that  $b = f(a^*)$ . Then  $p_j f(a^*) = p_j(b) = p_j f(a)$  and  $p_k f(a^*) = p_k(b) = p_k f(a')$  imply  $a \equiv a^*(q_j)$  and  $a^* \equiv a'(q_k)$ , i.e.,  $a \equiv a'(q_j \circ q_k)$ . In consequence  $\vee q_j = A^2$ . Let now  $\{a_j \mid j \in J\}$  be a family of elements in  $A$ . Consider  $f(a^*) = b \in \prod B_j$  such that  $p_j(b) = p_j f(a_j)$ . Then  $p_j f(a^*) = p_j(b) = p_j f(a_j)$ , i.e.,  $a^* \equiv a_j(q_j)$ ,  $j \in J$ .

Conversely, let be  $\mathcal{S} = \{q_j \mid j \in J\}$  such that conditions (i)-(iii) are satisfied. Consider the map  $f : A \rightarrow \prod A/q_j$  such that  $p_j f = \text{nat}_{q_j} : A \rightarrow A/q_j$ . The map  $f$  is injective:  $\ker(f) = \wedge(p_j f) = \wedge q_j = \Delta_A$ . For an element  $b \in \prod A/q_j$  let be  $a_j \in A$  such that  $p_j f(a_j) = p_j(b)$ . Taking into account (ii) and (iii) there exists  $a \in A$  such that  $a \equiv a_j(q_j)$  for all  $j \in J$ . Hence  $p_j f(a) = p_j f(a_j) = p_j(b)$ ,  $j \in J$  imply  $f(a) = b$ .  $\square$

### 3 Normal congruent families of equivalence relations

In this section, we collect some definitions and results that will be used later. For a more detailed exposition, the reader is referred to [2] and [3].

Let  $\mathcal{A} = (A, \alpha)$  and  $\mathcal{B} = (B, \beta)$  be quasigroups. A **homotopy**  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is an ordered system of maps  $\varphi = [f_1, \dots, f_n; f]$  from the set  $A$  to the set  $B$  such that

$$f\alpha(x_1, \dots, x_n) = \beta(f_1(x_1), \dots, f_n(x_n)) \quad (1)$$

for all  $x_1, \dots, x_n \in A$ .

The map  $f_i$ ,  $i \in \mathbb{N}_n = \{1, 2, \dots, n\}$  is known as the  $i$ -th component of  $\varphi$  and  $f$ -the principal component. The equality and composition of homotopies are defined componentwise.

The category **QGR** has the class of all quasigroups as its object class and its morphisms are quasigroup homotopies. Isomorphisms in **QGR** are called isotopies. They are just the homotopies having each component bijective.

The **kernel of homotopy**  $\varphi$  is  $\ker(\varphi) = [\ker(f_1), \dots, \ker(f_n); \ker(f)]$ .

A **normal congruent family of equivalences**  $\theta$  on a quasigroup  $\mathcal{A} = (A, \alpha)$  is an ordered system of equivalence relations on the set  $A$ ,  $\theta = [q_1, \dots, q_n; q]$ , such that for all  $a = (a_1, \dots, a_n) \in A^n$ .

$$T_i^2(q_i) = q, \text{ for all } i \in \mathbb{N}_n \quad (2)$$

where  $T_i : A \rightarrow A$ ,  $T_i(x) = \alpha(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$  is the  $i$ -th elementary transposition by  $a$ .

The kernel of homotopy  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a normal congruent family of equivalences on  $\mathcal{A}$ .

We show that the converse is also true.

Let  $\theta = [q_1, \dots, q_n, q]$  be a normal congruent family of equivalences on  $\mathcal{A} = (A, \alpha)$ . For any  $a = (a_1, \dots, a_n) \in A$   $T_i^* : A/q_i \rightarrow A/q$ ,  $T_i^*(q_i(x)) = q(T_i(x))$  is bijective for each  $i \in \mathbb{N}_n$  ( $q_i(x), q(x)$  – the blocks of  $x$ ).

We define an  $n$ -ary operation  $\bar{\alpha}$  on  $A/q$  by

$$\bar{\alpha}(q(x_1), \dots, q(x_n)) = q(\alpha(T_1^{-1}(x_1), \dots, T_n^{-1}(x_n))) \quad (3)$$

Then  $(A/q, \bar{\alpha})$  is a loop having  $e = \alpha(a_1, \dots, a_n)$  as a unit.

It is easy to see that  $\varphi = [f_1, \dots, f_n; f] : \mathcal{A} \rightarrow (A/q, \bar{\alpha})$  defined by  $f_i(x) = q(T_i(x))$ ,  $i \in \mathbb{N}_n$  and  $f(x) = q(x)$  is a homotopy and  $\ker(\varphi) = \theta$ .

The operation  $\bar{\alpha}$  depends on  $a$ . For an another element  $b = (b_1, \dots, b_n) \in A^n$  we obtain an another loop  $(A/q, \beta)$ . They are principal isotopic. So, the notation  $\mathcal{A}/\theta = (A/q, \alpha)$  is consistent. We call  $\mathcal{A}/\theta$  a **quotient quasigroup of  $\mathcal{A}$  by  $\theta$** .

For an  $n$ -quasigroup  $\mathcal{A}$ , let  $NCF(\mathcal{A})$  denote the set of normal congruent families of equivalences on  $\mathcal{A}$ . Define an order relation  $\leq$  on  $NCF(\mathcal{A})$  by setting

$$\theta_1 = [q_{11}, \dots, q_{n1}; q_1] \leq \theta_2 = [q_{12}, \dots, q_{n2}; q_2]$$

iff  $q_{i1} \subseteq q_{i2}$ ,  $i \in \mathbb{N}_n$  and  $q_1 \subseteq q_2$ .

If  $\mathcal{S} = \{\theta_j = [q_{1j}, \dots, q_{nj}; q_j] \mid j \in J\}$  is a family of normal congruent families of equivalences on  $\mathcal{A}$  then

$$\wedge \theta_j = [\wedge q_{1j}, \dots, \wedge q_{nj}; \wedge q_j]$$

and

$$\vee \theta_j = [\vee q_{1j}, \dots, \vee q_{nj}; \vee q_j]$$

are again normal congruent families of equivalences on  $\mathcal{A}$ . Thus  $NCF(\mathcal{A})$  forms a complete lattice under  $\leq$ .

Now let be  $\mathcal{S} = \{\theta_j \mid j \in J\} \subseteq NCF(\mathcal{A})$ . Then  $\mathcal{S}_i = \{q_{ij} \mid j \in J\}$ ,  $i \in \mathbb{N}_n$  and  $\mathcal{S}_{n+1} = \{q_j \mid j \in J\}$  are families of equivalence relations on  $A$  called **components of  $\mathcal{S}$** . By (2), if one component of  $\mathcal{S}$  is absolutely (weakly) permutable then all components are absolutely (weakly) permutable.

We call  $\mathcal{S}$  **absolutely (weakly) permutable** if all its components are absolutely (weakly) permutable.

## 4 Direct decompositions in QGR

We present the homotopic properties of direct products of quasigroups.

Let  $\{\mathcal{A}_j = (A_j, \alpha_j) \mid j \in J\}$  be a family of quasigroups. The direct product of this family is the quasigroup  $\prod \mathcal{A}_j = (\prod A_j, \alpha)$  whose underlying set is the cartesian product  $\prod A_j$  and operation  $\alpha$  is defined coordinatewise. The projections  $p_j : \prod \mathcal{A}_j \rightarrow \mathcal{A}_j$ ,  $j \in J$ ,  $p((a_j)_{j \in J}) = a_j$ ,  $j \in J$  are quasigroup homomorphisms.

**Theorem 4.** *The category **QGR** has products.*

*Proof.* Let  $(\prod \mathcal{A}_j, \{p_j \mid j \in J\})$  is a product in **QGR**. Indeed, let be  $\varphi_j = [f_{1j}, \dots, f_{nj}; f_j] : \mathcal{B} \rightarrow \mathcal{A}_j$ ,  $j \in J$ . Consider the maps  $f_i, f : \mathcal{B} \rightarrow \prod \mathcal{A}_j$  defined by  $p_j f_i = f_{ij}$ ,  $i \in \mathbb{N}_n$  and  $p_j f = f_j$ ,  $j \in J$ . It is easy to show that  $\varphi = [f_1, \dots, f_n; f] : \mathcal{B} \rightarrow \prod \mathcal{A}_j$  is the unique homotopy with  $p_j \varphi = \varphi_j$ ,  $j \in J$ .  $\square$

Let  $\mathcal{A}$  be a quasigroup and let  $\{\mathcal{A}_j \mid j \in J\}$  be a family of quasigroups.

**Definition 1.** *A decomposition of  $\mathcal{A}$  as a **direct product of  $\{\mathcal{A}_j \mid j \in J\}$  is a **QGR-isomorphism (quasigroup isotopy)  $\varphi : \mathcal{A} \rightarrow \prod \mathcal{A}_j$ . The decomposition is called **proper** if none of the homotopies  $p_j \varphi$  is a **QGR-monomorphism (a homotopy with all component injective)**.  $\mathcal{A}$  is called **direct indecomposable** if it admits no proper direct decomposition.*****

**Theorem 5.**  *$\mathcal{A}$  has a proper direct decomposition iff there exists a family  $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$  such that:*

- (i)  $\wedge \theta_j = \Delta_A$ ;
- (ii)  $\vee \theta_j = A^2$ ;
- (iii)  $\mathcal{S}$  is absolutely permutable.

*Proof.* Let  $\varphi = [f_1, \dots, f_n] : \mathcal{A} \rightarrow \prod \mathcal{A}_j$  be a proper direct decomposition. Put  $\theta_j = \ker(p_j \varphi)$ ,  $j \in J$ . Taking into account Theorem 3 it is easy to show that  $\mathcal{S} = \{\theta_j \mid j \in J\}$  satisfies conditions (i) – (iii).

Conversely, suppose that  $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$  satisfies conditions (i) – (iii). It is easy to see that all its components satisfy conditions (i) – (iii). Consider the direct product  $\prod \mathcal{A}/\theta_j$  and the homotopy

$$\varphi = [f_1, \dots, f_n; f] : \mathcal{A} \rightarrow \prod \mathcal{A}/\theta_j$$

defined by  $p_j \varphi = \varphi_j$  where  $\varphi_j : \mathcal{A} \rightarrow \mathcal{A}/\theta_j$  are the canonical homotopies defined in previous section.

By Theorem 3 it follows that  $\varphi$  is a proper direct decomposition of  $\mathcal{A}$ .  $\square$

By Theorem 5 and Theorem 1 we get

**Theorem 6.**  $\mathcal{A}$  has a proper direct decomposition iff there exists a family  $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$  such that:

- (i)  $\wedge \theta_j = \Delta_A$ ;
- (ii)  $\theta_j \vee \theta_k = A^2$ , for any  $\theta_j, \theta_k \in \mathcal{S}$ ,  $\theta_j \neq \theta_k$ ;
- (iii)  $\mathcal{S}$  is weakly permutable.

By Theorem 5 and Theorem 2 we get

**Theorem 7.** (Chinese remainder theorem).  $\mathcal{A}$  has a finite proper direct decomposition iff there exists a finite family  $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\} \subseteq NCF(\mathcal{A})$  such that:

- (i)  $\wedge \theta_j = \Delta_A$ ;
- (ii)  $\theta_j \circ \bar{\theta}_j = A^2$ ,  $j \in J$ .

The following theorem is useful to characterize direct indecomposable quasigroups.

**Theorem 8.**  $\mathcal{A}$  has a proper direct decomposition iff there exists  $\theta_1, \theta_2 \in NCF(\mathcal{A})$  such that:

- (i)  $\theta_1, \theta_2 > \Delta_A$ ,  $\theta_1 \neq \theta_2$ ;
- (ii)  $\theta_1 \wedge \theta_2 = \Delta_A$ ;
- (iii)  $\theta_1 \circ \theta_2 = A^2$ .

*Proof.* Suppose that  $\mathcal{A}$  has a proper direct decomposition. Let be  $\mathcal{S} = \{\theta_j > \Delta_A \mid j \in J\}$  as in Theorem 5. There exists  $\theta_i \in \mathcal{S}$  such that  $\theta_i < A^2$ . Then  $\bar{\theta}_i > \Delta_A$ , and  $\theta_i \wedge \bar{\theta}_i = \Delta_A$ . By Theorem 2,  $\theta_i \circ \bar{\theta}_i = A^2$ .

The converse follows by Theorem 7. □

**Corollary 1.**  $\mathcal{A}$  is direct indecomposable iff there is no pair  $\theta_1, \theta_2 \in NCF(\mathcal{A})$ ,  $\theta_1 \neq \theta_2$  with

- (i)  $\theta_1, \theta_2 > \Delta_A$ ;
- (ii)  $\theta_1 \wedge \theta_2 = \Delta_A$ ;
- (iii)  $\theta_1 \circ \theta_2 = A^2$ .

A quasigroup direct indecomposable in the subcategory **Qgr** (whose morphisms are quasigroup homomorphisms) of **QGR** can be proper decomposable in **QGR**.

Example. Let  $\mathcal{A} = (A, \cdot)$  be the binary quasigroup.

	1	2	3	4	5	6	7	8
1	3	4	6	7	1	2	5	8
2	4	3	7	6	2	1	8	5
3	7	6	4	3	8	5	2	1
4	6	7	3	4	5	8	1	2
5	1	2	5	8	3	4	6	7
6	2	1	8	5	4	3	7	6
7	8	5	2	1	7	6	4	3
8	5	8	1	2	6	7	3	4

It is easy to see that only  $\Delta_A$  and  $A^2$  are normal congruences on  $\mathcal{A}$ . Hence  $\mathcal{A}$  is direct indecomposable in  $Qgr$ .  $\mathcal{A}$  is direct indecomposable in  $\mathbf{Qgr}$ , but  $\mathcal{A}$  has a proper direct decomposition in  $\mathbf{QGR}$ :  $\theta = [q_1, q_2; q]$  defined by

$$\begin{aligned} A/q_1 &= \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}\} \\ A/q_2 &= \{\{1, 4\}, \{2, 3\}, \{5, 8\}, \{6, 7\}\} \\ A/q &= \{\{1, 8\}, \{2, 5\}, \{3, 7\}, \{4, 6\}\} \end{aligned}$$

and  $\theta' = [q'_1, q'_2; q']$  defined by

$$\begin{aligned} A/q'_1 &= A/q'_2 = \{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\} \\ A/q' &= \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\} \end{aligned}$$

verify conditions of Theorem 8.

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