ISOGONAL TRANSFORMATIONS REVISITED WITH GEOGEBRA

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Abstract: The symmedian lines and the symmedian point of a given triangle present interesting properties. Part of these properties can be formulated in a more general context for isogonals.

In a triangle the isogonal of a line passing through one of the vertices of the triangle is a line symmetric to the bisector of the given angle.

It can be proven that the three isogonals of three concurrent lines which pass through the three vertices of the triangle, are concurrent as well.

This property serves as definition for the isogonal transformation, the image of a given point in this transformation will be the intersection point of the three isogonals of the three lines which pass through the given point and the vertices of the triangle.

The paper is aimed to present some of the properties of the isogonal transformations, and to visualise them using GeoGebra.

Keywords: symmedian, isogonal lines, isogonal transforms, trilinear coordinates.

The symmedian line in a triangle ABC is the symmetric of a median of the triangle, the three symmedians are meeting in the symmedian point K of the triangle, this point is the isogonal transform of the centroid G. Similar to the definition of the symmedian lines one can introduce the isogonal of any line in a triangle. The isogonals of the so called Cevian lines, the three lines which join the three vertices of a triangle with a given point P of a triangle, will be concurrent in a point P', then is the isogonal transform of P. The early research papers on isogonals and trilinear coordinates include [1], [2], [3], and [4].

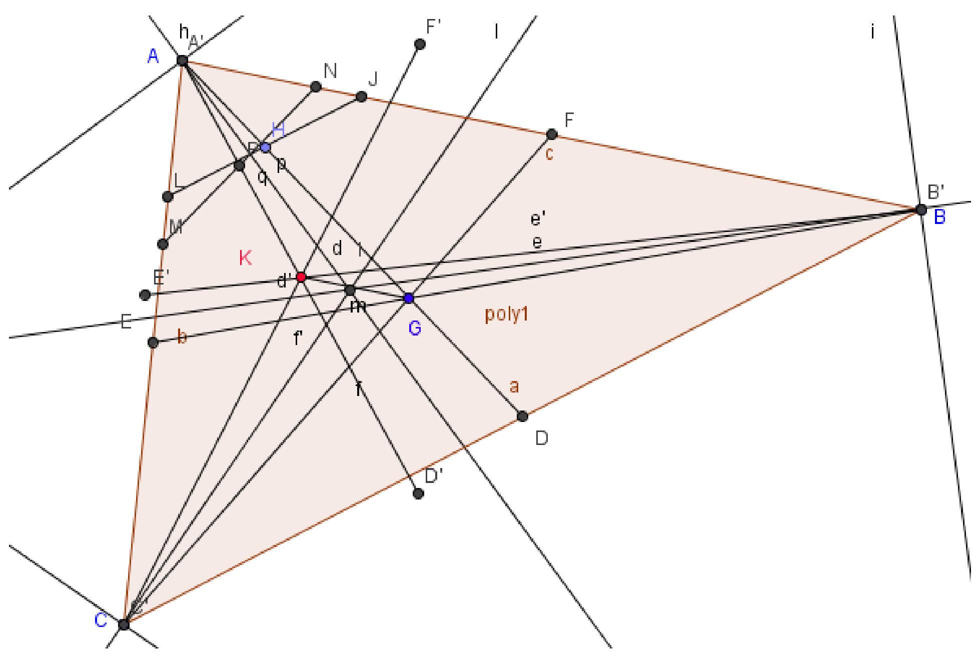


Figure 1. The centroid and the symmedian point of a triangle

A useful possibility to study this properties is based on the use of trilinear coordinates. It can be proven that if (α, β, γ) is the trilinear coordinate triplet of a point P, then $\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$ is the coordinate triplet of its isogonal conjugate. It is easy to prove e.g. that the trilinear coordinates of the excentre and the ortocentre of given triangle satisfy the given conditions, hence the two point are isogonal conjugated.

Menelaos' Theorem

Let us consider a triangle ABC and a line, which intersects the sides BC, CA and AB respectively in the points F, G and H then the following relation is true:

$$\frac{FB}{FC}\frac{GC}{GA}\frac{HA}{HB} = 1$$

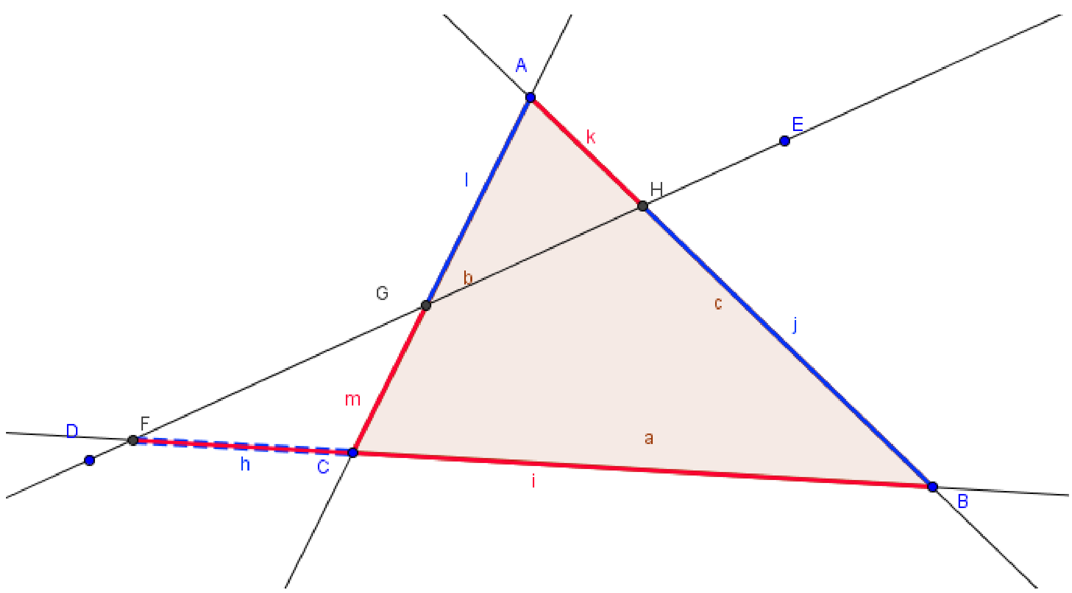


Figure 2. The Menelaos' Theorem

Remark: The inverse theorem is also true.

Ceva's Theorem

Let us consider a triangle ABC and three lines, and the points F, G and H on the sides AF, BG and CH respectively. The three lines are concurrent in a point P exacly when the following relation is true:

$$\frac{FB}{FC}\frac{GC}{GA}\frac{HA}{HB} = -1$$

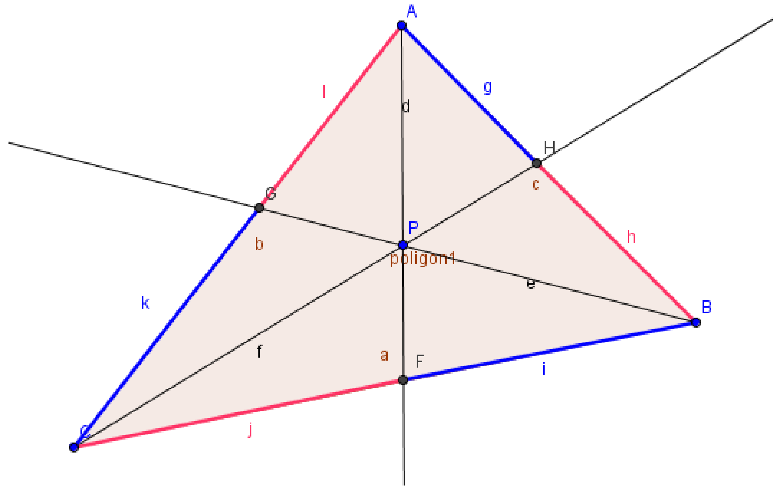


Figure 3. The Ceva's Theorem

Remark: The inverse theorem is also true.

The trigonometric form of Ceva's Theorem

Let us consider a triangle ABC and three lines, and the points F, G and H on the sides AF, BG and CH respectively. The three lines are concurrent in a point P exactly when the following relation is true:

$$\frac{\sin \alpha_1}{\sin \alpha_2} \frac{\sin \beta_1}{\sin \beta_2} \frac{\sin \gamma_1}{\sin \gamma_2} = 1$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2,$ and γ_1, γ_2 respectively are the angles determinded by the three lines at the vertices A, B and C.

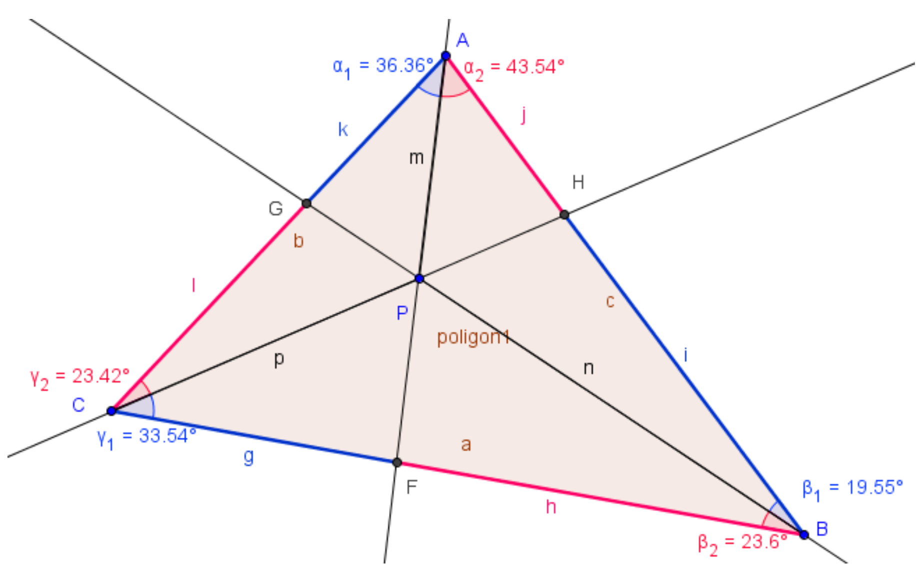


Figure 4. The trigonometric form of the Ceva's Theorem

Proof of the trigonometric form:

Denote the angles of the triangle ABC by α, β and γ respectively, and denote by $\alpha_1, \alpha_2, \beta_1, \beta_2,$ and γ_1, γ_2 the respective angles at the vertices A, B and C.

Let us apply the sinus law for the three pairs of two-by two triangles got by "slicing" the original triangle in two by the given lines (eg. $\Delta\ ABM$ and $\Delta\ MAC$).

We get the following six relations:

$$\frac{BF}{AF} = \frac{\sin\alpha_2}{\sin\beta}; \frac{CF}{AF} = \frac{\sin\alpha_1}{\sin\gamma}; \frac{CG}{BG} = \frac{\sin\beta_2}{\sin\gamma}; \frac{AG}{BG} = \frac{\sin\alpha_1}{\sin\alpha}; \frac{AH}{CH} = \frac{\sin\gamma_2}{\sin\alpha}; \frac{BH}{CH} = \frac{\sin\gamma_1}{\sin\beta}$$

hence: $\frac{BF}{CF} = \frac{\sin\alpha_2}{\sin\beta}\frac{\sin\gamma}{\sin\alpha_1}; \frac{CG}{AG} = \frac{\sin\beta_2}{\sin\gamma}\frac{\sin\alpha}{\sin\beta_1}$ and $\frac{AH}{BH} = \frac{\sin\gamma_2}{\sin\alpha}\frac{\sin\beta}{\sin\gamma_1}$, now introducing them in the Ceva"s theorem: $\frac{BF}{CF}\frac{CG}{AG}\frac{AH}{BH} = 1$, we get $\frac{\sin\alpha_2}{\sin\beta}\frac{\sin\gamma}{\sin\alpha_1}\frac{\sin\beta_2}{\sin\gamma}\frac{\sin\alpha}{\sin\beta_1}\frac{\sin\gamma_2}{\sin\alpha}\frac{\sin\beta}{\sin\gamma_1} = 1$, which ends the proof after simplifying it.

In a triangle the isogonal of a line passing through one of the vertices of the triangle is a line symmetric to the bisector of the given angle. It can be proven that the three isogonals of three concurrent lines which pass through the three vertices of the triangle, are concurrent.

The isogonal conjugate of a point ***P*** of the triangle ABC is the intersection ***P'*** of the isogonals AP', BP' and CP' of the three lines AP, BP and CP.

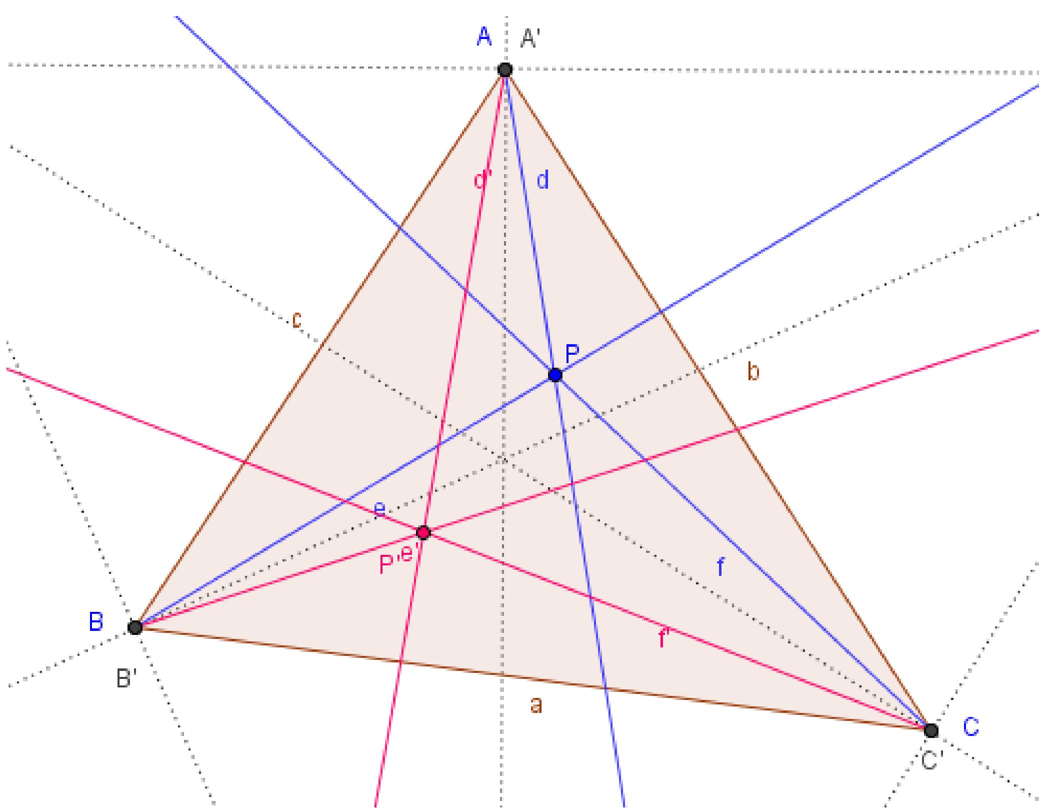


Figure 5. The isogonal transform P' of a point P in a triangle

The fixpoint of the isogonal transformation is the incentre (the centre of the incircle, i.e. the intersection of the three bisectors).

The position of the isogonal conjugate points *P* and *P'* can be described with the so called trilinear coordinates.

The trilinear coordinates of the point *P* is a triplet (α, β, γ), such as the triple ratio $\alpha : \beta : \gamma$ equal to the triple ratio $x_a : x_b : x_c$ where x_a, x_b, x_c represent the distances of the point P from the respective sides of the triangle, in other words, there is a $k \neq 0$, such that: $\alpha = kx_a, \beta = kx_b, \gamma = kx_c$.

It can be proven that if the trilinear coordinates of *P* are $P = (\alpha, \beta, \gamma)$, then $P' = \left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$.

Proof: Let us denote as by x_a, x_b, x_c the distances of the point P from the sides, and similarly, by y_a, y_b, y_c the distances of the isogonal conjugate P' from the sides.

As the two triangles, $\Delta\ APE$ and $\Delta\ AP'I$ are similar, we have $\frac{x_c}{y_b} = \frac{AP}{AP'}$, moreover we have from two other similar triangles: $\Delta\ APF\ \Delta\ AP'G$ the relation: $\frac{x_b}{y_c} = \frac{AP}{AP'}$, hence $\frac{x_b}{y_c} = \frac{x_c}{y_b}$, so $\frac{x_b}{x_c} = \frac{\frac{1}{y_b}}{\frac{1}{y_c}}$, finally $\frac{x_b}{\frac{1}{y_b}} = \frac{x_c}{\frac{1}{y_c}}$.

The proof can be repeated circularly for the other pairs of sides, hence

$$\frac{x_a}{\frac{1}{y_a}} = \frac{x_b}{\frac{1}{y_b}} = \frac{x_c}{\frac{1}{y_c}}.$$

Several remarcable points of the triangle isogonally conjugated, e.g. the ortocentre and the centre of the excircle, or the first and second Brocard points of the triangle are such pairs or isogonal conjugates.

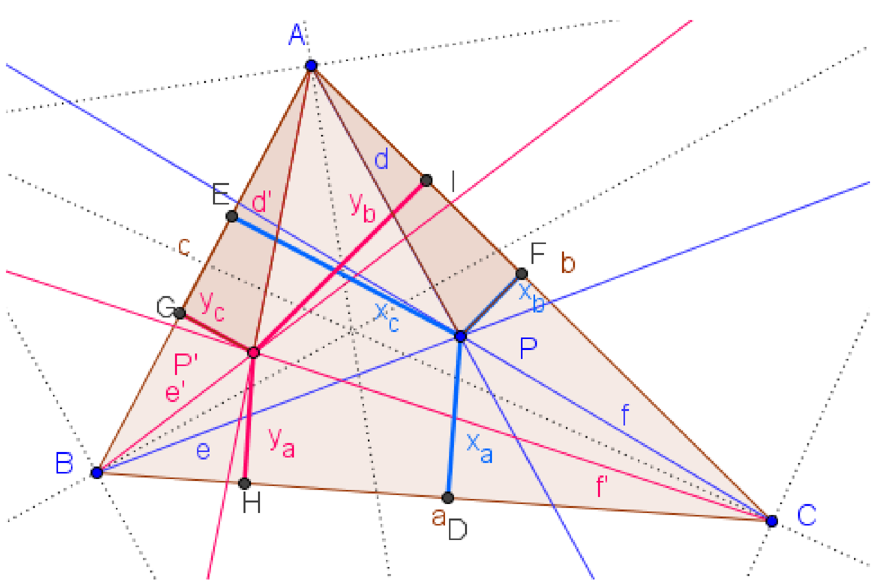


Figure 6. The trilinear coordinates of the isogonal transforms P and P'

Let us see now the trilinear coordinates $P(x,y,z)$ for some of the remarcable points of the triangle.

1. The centre *I* of the incircle is $I(1,1,1)$
2. The centre ***O*** of the circumcircle is $\boldsymbol{O}(\cos\alpha, \cos\beta, \cos\gamma)$.

Proof: in $\Delta\ COF$ the measure of the angle COF is 2α, hence $x_a = OF = R\cos\alpha$, similarly $x_b = R\cos\beta, x_c = R\cos\gamma$, where R is radius of the excircle.

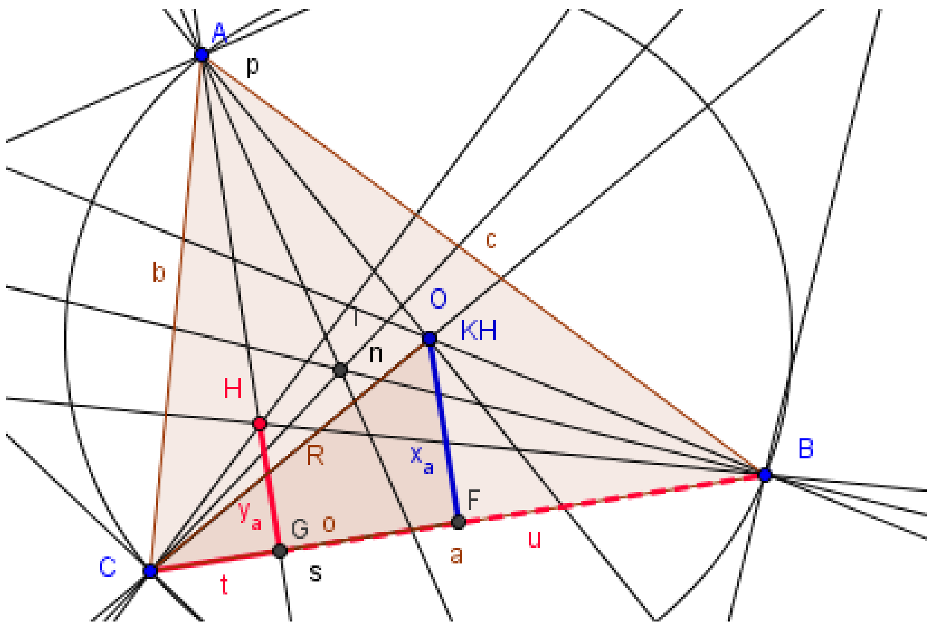


Figure 7. The excentre O and ortocentre H are isogonal transforms

3. The ortocentre ***H*** of the triangle is ***H*** $(\sec\alpha, \sec\beta, \sec\gamma)$.

Proof: in $\Delta\ \boldsymbol{HGC}$ and in $\Delta\ \boldsymbol{HGB}$ $y_a = t\cot\beta$, and $y_a = u\cot\gamma$, where $t = CG$ and $u = \boldsymbol{BG}$, moreover $t = b\cos\gamma$ (in $\Delta\ \boldsymbol{AGC}$) and $u = c\cos\beta$ (in $\Delta\ \boldsymbol{AGB}$), consequently we have two ways to express: $y_a = b\cos\gamma\cot\beta$ and $y_a = c\cos\beta\ \cot\gamma$. Similarly $y_b = a\cos\gamma\cot\alpha$ and $y_b = c\cos\alpha\cot\gamma$.

The ratio $\frac{y_a}{y_b} = \frac{c\cos\beta\cot\gamma}{c\cos\alpha\cot\gamma} = \frac{\cos\beta}{\cos\alpha}$, in other words: $\frac{y_a}{y_b} = \frac{\sec\alpha}{\sec\beta}$, and the steps can be repeated by circular permutation for the other two pairs, y_b, y_c and y_c, y_a. This means $\boldsymbol{H} = (\sec\alpha, \sec\beta, \sec\gamma)$ indeed.

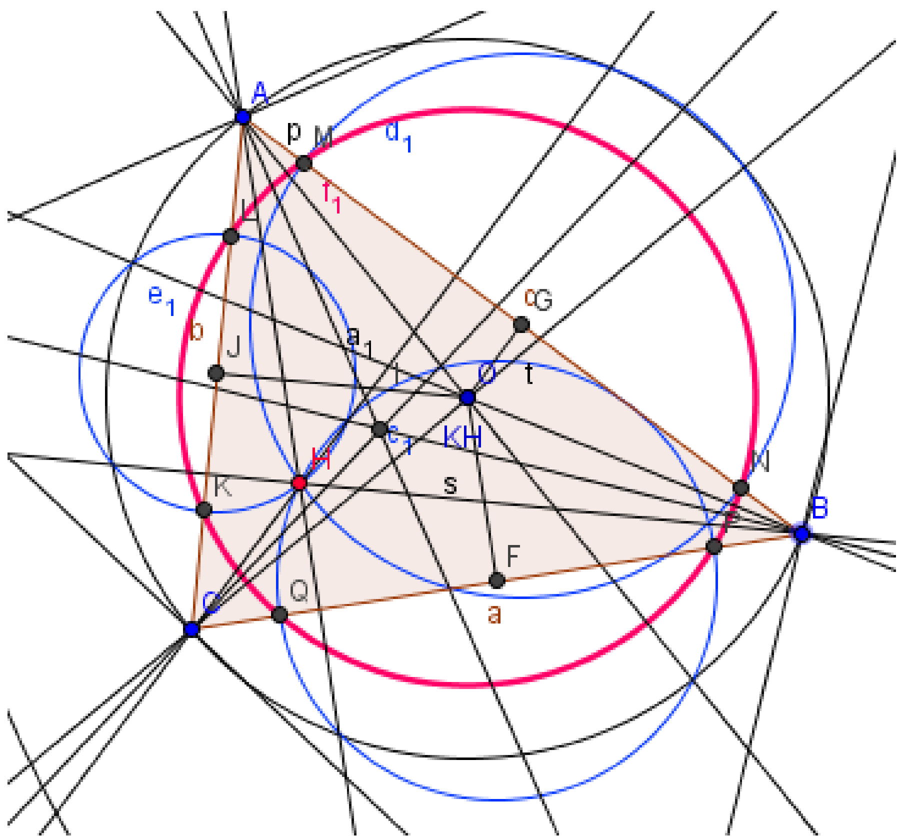
Conclusion: we have proven that the circumcentre ***O*** and the ortocentre ***H*** are isogonal conjugates.

Applications

The isogonal transforms are subject of several recent recearch papers see e.g. [5], [6]. However, we will cite only two didactical applications:

IMO problem 2008/1. [7]

Let $\boldsymbol{H}$ be the orthocenter of an acute-angled triangle $\boldsymbol{ABC}$. The circle Γ_A centered at the midpoint of $\boldsymbol{BC}$ and passing through $\boldsymbol{H}$ intersects the sideline $\boldsymbol{BC}$ at points $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$. Similarly, define the points $\boldsymbol{B_1, B_2, C_1}$ and $\boldsymbol{C_2}$. Prove that six points $\boldsymbol{A_1, A_2, B_1, B_2, C_1}$ and $\boldsymbol{C_2}$ are concyclic.



Generalisation by R. Warrendorf [8]

Let $\boldsymbol{ABC}$ be a triangle. Let $\boldsymbol{P}$ be a point and $\boldsymbol{P'}$ the isogonal conjugate of $\boldsymbol{P}$ with respect to $\boldsymbol{ABC}$. Let $\boldsymbol{R, S}$, and $\boldsymbol{T}$ be the orthogonal projections of $\boldsymbol{P}$ onto $\boldsymbol{AB, AC}$, and $\boldsymbol{BC}$.

Let the circles with centers at $\boldsymbol{R, S}$, and $\boldsymbol{T}$ and passing through $\boldsymbol{P'}$ intersect the sidelines of $\boldsymbol{BC, AC}$, and $\boldsymbol{AB}$ at A', A'', B', B'', C', and C''. Then A', A'', B', B'', C', and C'' are concyclic.

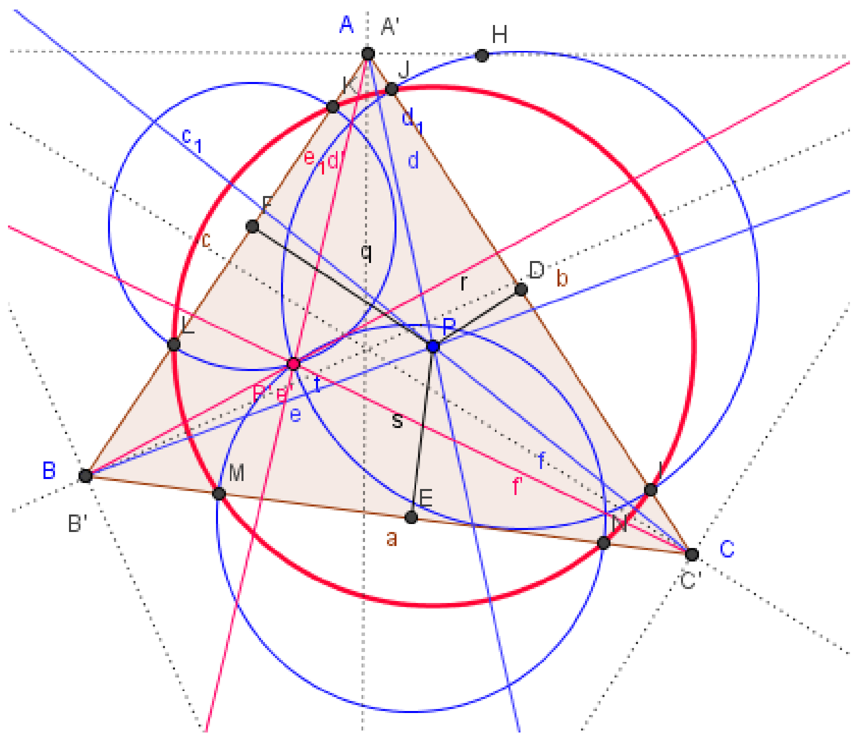


Figure 9. The generalisation of the problem 1. IMO 2008

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