THE FREQUENCY SPECTRUM FOR THE LOCAL FOURIER ANALYSIS OF THE MULTIGRID METHOD IN THE ONE-DIMENSIONAL CASE

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Abstract: In this article we study a convection-diffusion problem in the one dimensional case from the point of view of the effect of an iterative numerical method on the components of the error. We propose here a new definition for the smoothing factor, and a new way to split the frequency spectrum used to describe the error term in the local Fourier analysis of the multigrid method.

Keywords: local Fourier analysis, frequencies spectrum , multigrid, amplification factor, smoothing factor

Introduction

For the study of iterative numerical methods, one of the most efficient and frequently used methods is the local Fourier analysis (LFA). This method is used in order to compute the amplification factor, the smoothing factor, the error reduction factor and the convergence factor for a numerical method. The computing of the smoothing factor is very important because it's analysis allows designing efficient components for the multigrid method, being well known that this method has to be adapted to each type of problem that has to be solved. The LFA, first introduced by A. Brandt in [3], then used in [1], [2] has become a widely used method [4], [11], [12] on a large variety of problems.

The novelty in this paper is the definition that we propose for the smoothing factor- used for both the design and analysis of the multigrid method, and the new way to split the frequency spectrum into high and low frequencies for the one dimensional case.

The model problem used here is the mathematical representation of the stationary convection-diffusion process:

$$\begin{cases} -\varepsilon u''(x)+au'(x)=f(x), & x\in\Omega=(0,1), \\ u(x)=0, & x\in\Gamma=\{0,1\}, \end{cases} \qquad \textbf{(1)}$$

where u is the concentration of the substance, f is a possible perturbation term for the concentration, due for example to chemical reactions, ε is the diffusion coefficient and a- the convection coefficient.

In order to numerically solve a differential equation, it is discretized, using for example the finite differences method of second order. This process leads to the following system:

$$\begin{cases} 4\varepsilon v_1-(2\varepsilon-ah_l)v_2=h_l^2 f_1, \\ -(2\varepsilon+ah_l)v_{j-1}+4\varepsilon v_j-(2\varepsilon-ah_l)v_{j+1}=h_l^2 f_j, & j=2,\ldots,n_l-1. \\ -(2\varepsilon+ah_l)v_{n_l-1}+4\varepsilon v_{n_l}=h_l^2 f_{n_l}, \end{cases} \qquad \textbf{(2)}$$

As the exact solution u of the differential equation is also a solution of the system (2), the error $e = u - v$ will satisfy the system:

$$\begin{cases} 4\varepsilon e_1 - (2\varepsilon - ah_l)e_2 = 0_1, \\ -(2\varepsilon + ah_l)e_{j-1} + 4\varepsilon e_j - (2\varepsilon - ah_l)e_{j+1} = 0, \qquad j = 2,..., n_l - 1. \\ -(2\varepsilon + ah_l)e_{n_l-1} + 4\varepsilon e_{n_l} = 0, \end{cases} \quad \textbf{(3)}$$

If the system (3) is solved using:

- the Gauss-Seidel method, then the iterate after *m* steps can be computed from the relation:

$$-(2\varepsilon + ah_l)e_{j-1}^{(m)} + 4\varepsilon e_j^{(m)} = (2\varepsilon - ah_l)e_{j+1}^{(m-1)}, \qquad m \geq 1,\ j = 2,..., n_l - 1; \quad \textbf{(4)}$$

- the Pondered Jacobi method, the iterate at the step *m* is obtained from the relations:

$$4\varepsilon\, e^* = (2\varepsilon + ah_l)e_{j-1}^{(m-1)} + (2\varepsilon - ah_l)e_{j+1}^{(m-1)} \quad \textbf{(5)}$$

and

$$e_j^{(m)} = \omega e^* + (1-\omega)e_j^{(m-1)} \quad \textbf{(6)}$$

where $\omega \in (0,1), m \geq 1,\ j = 1,..., n_l - 1$ and $e_j^{(m)}$ is the error value after m iterations in the point $x_j,\ j = 0,..., n_l$.

As the error is a vector with n_l components, the value of it in a point x_j can be expanded in a discrete Fourier series [9], for example using the base: $\left\{ e^{it_k} \,\middle|\, t_k = \frac{2\pi k}{1+n_l}, \quad j,k = 0,1,..., n_l \right\}$, with the formula:

$$e_j^{(m)} = \sum_{k=0}^{n_l} c_k^{(m)} e^{it_k},\ j = 0,..., n_l. \quad \textbf{(7)}$$

From the system (5), for the Gauss-Seidel method the relation between two successive iterations can be written as:

$$-(2\varepsilon + ah_l)\sum_{k=0}^{n_l} c_k^{(m)} e^{i(j-1)t_k} + 4\varepsilon \sum_{k=0}^{n_l} c_k^{(m)} e^{ijt_k} = \sum_{k=0}^{n_l} (2\varepsilon - ah_l)c_k^{(m-1)} e^{i(j+1)t_k}.$$

From this relation it follows that:

$$\sum_{k=0}^{n_l} e^{ijt_k} \{ c_k^{(m)} \left[4\varepsilon - (2\varepsilon + ah_l)e^{-it_k} \right] - c_k^{(m-1)} (2\varepsilon - ah_l)e^{it_k} \} = 0,\ j = 0,..., n_l. \quad \textbf{(8)}$$

As the vectors e^{ijt_k} are orthonormal:

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi k}{N}m} e^{i\frac{2\pi k}{N}n} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} = \delta_{m,n}, \quad \textbf{(9)}$$

the equality (8) is true for any $j = 0,..., n_l$ if:

$$c_k^{(m)} \left[4\varepsilon - (2\varepsilon + ah_l)e^{-it_k} \right] - c_k^{(m-1)} (2\varepsilon - ah_l)e^{it_k} = 0, \quad k = 0,..., n_l. \quad \textbf{(10)}$$

The amplification factor and the smoothing factor of a numerical method

In order to measure the growth or the decrease of a Fourier mode during one iterative step and the convergence speed of a numerical method, one can use the amplification factor as defined in [5], [12]:

Definition 1. *For a numerical iterative method,* ***the amplification factor*** $g(t_k)$ *is the ratio between the coefficient* $c_k^{(m)}$ *after* m *iterates and the coefficient from the previous iterative step,* $c_k^{(m-1)}$.

The convergence speed of a numerical method is better when the module of the amplification factor smaller than 1.

The amplification factor of the Gauss-Seidel method can be obtained from (10):

$$g(t_k) = \frac{(2\varepsilon - ah_l)e^{it_k}}{4\varepsilon - (2\varepsilon + ah_l)e^{-it_k}}, k = 0, \ \ldots \ , n_l, \qquad \textbf{(11)}$$

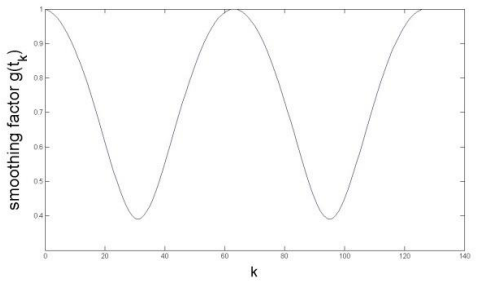
with the module:

$$|g(t_k)| = \frac{|2\varepsilon - ah_l|}{\sqrt{(2\varepsilon + ah_l)^2 + (4\varepsilon)^2 - 8\varepsilon(2\varepsilon + ah_l)\cos(t_k)}}, \quad k = 0,\ldots,n_l \qquad \textbf{(12)}$$

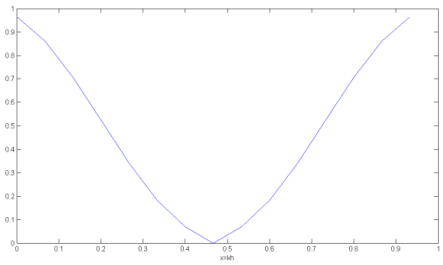
so that:

$$max\{|g(t_k)|, k = 0,\ldots,n_l\} = |g(0)| = 1$$

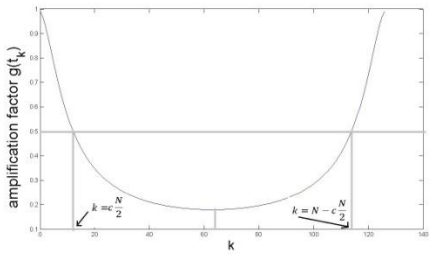
and has the graphic showed in (**Figure 1c**).



(a) Jacobi method



(b) pondered Jacobi method (ω=0.5)



(c) Gauss-Seidel method

Figure 1: The module of the amplification factor for $l = 6, \varepsilon = 0.1$ and $a = 10$ in the 1-dimensional case

For the pondered Jacobi method (**Figure1b**):

$$g(t_k)=\frac{4\varepsilon(1-\omega)+\omega(2\varepsilon+ah_l)e^{it_k}+\omega(2\varepsilon-ah_l)e^{-it_k}}{4\varepsilon},k=0,\ \ldots\ ,n_l \qquad \textbf{(13)}$$

and

$$|g(t_k)|=\sqrt{(1-\omega+\omega\cos t_k)^2+\left(\frac{ah\omega}{2\varepsilon}\sin t_k\right)^2}. \qquad \textbf{(14)}$$

When the boundary conditions are periodic, "the solution of the equation (2) is only determined up to a constant, so the Fourier mode with $k=0$ does not have to decrease during an iteration" [12], so that the measure used for convergence is:

$$max\{|g(t_k)|,k=1,...,n_l\}.$$

Remark 1. *In* ***Figures 1b*** *and* ***1c*** *it can be seen that for the frequencies* $k\in\left[c\frac{N}{2},N-c\frac{N}{2}\right],c\in(0,1)$ *the amplification factor has the module* $|g(t_k)|\leq\frac{1}{2}$. *This means that for the error components that have these frequencies, the numerical iterative method is efficient: in the worst case they are reduced by a factor of at least* $\frac{1}{2}$ *on each iteration step. On the other hand, for the components having frequencies between* $\left(0,c\frac{N}{2}\right)\cup\left(N-c\frac{N}{2},N\right)$, *the amplification factor is almost 1, thus for these frequencies the method is not efficent: these components remain almost unchanged after one iterative step.*

A well known property of a numerical iterative method used to solve a linear system is the fact that it reduces efficiently the components that are oscillatory. One way to measure this smoothing property is to determine the smoothing factor.

Definition 2 [11]. *The* ***smoothing factor*** *of a numerical iterative method having the iteration matrix* M *is the worst amplification factor module, taken for all the high frequencies and is denoted by* $\rho(M)=\max\{|g(t_k)|,t_k\in T_{high}\}$.

level	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
$\rho(M_J)$	1	1	1	1	1	1
$\rho(M_{\omega J})$	0	0.5	0.8536	0.9619	0.9904	0.9976
$\rho(M_{GS})$	0.3333	0.4472	0.6786	0.8756	0.9637	0.9905

Table 1. The smoothing factor of Jacobi (*J*), Jacobi pondered (ωJ) and Gauss-Seidel (GS) methods for problem (2), $a=0,\varepsilon=1$

The data from **Table 1** show that the convergence speed of the studied methods decreases as the grid step becomes smaller due to the poor reduction of the low frequencies. Thus, these numerical methods are slowly convergent.

One of the most efficient methods to overcome this disadvantage of a classical iterative method is the multigrid method, which combines the property of such a method to reduce the high frequencies with the coarse grid correction method that has complementary properties: it reduces well the low frequencies.

The low and high frequencies spectrum

In order to design the multigrid method and for the study of its convergence and error reduction properties it is necessary to split the frequency spectrum into high and low frequencies. The splitting is made according to the effect of the smoothing method on the error components: the low frequencies are reduced very little or not at all by the numerical method, while the high frequencies are efficiently reduced, this being the fundamental property of a classical numerical method.

W. Hackbush showed in [5] that for problem (2) with $a=0$ and $\varepsilon=1$, the smoothing factor of the pondered Jacobi method is:

$$\rho(M_{\omega J})=\max_k\left\{1-2\omega\sin^2\frac{k\pi h_l}{2}, k=1,...,n_l, h_l=1/(n_l+1)\right\}. \qquad \textbf{(15)}$$

In order to determine this factor he determined the eigenvectors: $v_k=\sqrt{2h_l}(\sin kj\pi h_l)_{j=1}^{n_l}, k=1,...,n_l$ and eigenvalues: $\lambda_k=4h_l^{-2}\sin^2\frac{k\pi h_l}{2}, k=1,...,n_l$ of the matrix of the system obtained after the discretization process:

$$L_l=\frac{1}{h_l^2}\begin{pmatrix} 2 & -1 & 0 & 0 & ... & 0\\ -1 & 2 & -1 & 0 & ... & 0\\ 0 & -1 & 2 & -1 & ... & 0\\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & ... & 2 \end{pmatrix}.$$

For the pondered Jacobi method, the relation between two successive error iterates is: $e^{(m)}=e^{(m-1)}-\omega D_l^{-1}L_l e^{(m-1)}$, where D_l is the diagonal part of the matrix. If each term of the error is written with it's corresponding eigenvectors and eigenvalues:

$$\sum_{k=1}^{n_l}c_k^{(m)}v_k=\sum_{k=1}^{n_l}c_k^{(m-1)}v_k-\omega D^{-1}L\sum_{k=1}^{n_l}c_k^{(m-1)}v_k=\sum_{k=1}^{n_l}c_k^{(m-1)}v_k-\omega D^{-1}\sum_{k=1}^{n_l}c_k^{(m-1)}\lambda_k v_k,$$

then the amplification factor is:

$$g(t_k)=\frac{c_k^{(m)}}{c_k^{(m-1)}}=1-2\omega\sin^2\frac{k\pi h_l}{2}.$$

Analysing the values obtained for the convergence rate, W. Hackbush defined the high frequencies as the ones with $k\in\left[\frac{n_l+1}{2}, n_l\right]$ and the low frequencies: those with $k\in\left[1,\frac{n_l+1}{2}\right]$.

For the same problem, using the LFA method as in [12], the smoothing factor for the pondered Jacobi method ((14) with $a=0,\varepsilon=1$) is:

$$\rho(M_{\omega J})=\max_k\{|g(t_k)|,k=1,...,n_l\}=\max_k\left\{\left|\frac{c_k^{(m)}}{c_k^{(m-1)}}\right|,k=1,...,n_l\right\}= \\ =\max_k\{1-2\omega\sin^2(k\pi h_l),k=1,...,n_l\}. \quad \textbf{(16)}$$

For the smoothing factor obtained with the local Fourier analysis method ((14) or (16)), although very close to the convergence rate form (15), the definition of low and high frequencies cannot be done like in [5] because the graphic of the amplification factor (**Figure1b**) is symmetric with respect to the $k=\frac{N}{2}$ vertical line. For this reason and taking into account the **Remark 1**, when using the local Fourier analysis of a numerical iterative method or the multigrid method we propose the following definition of the smoothing factor:

Definition 3. *The* ***smoothing factor*** *of an iterative numerical method that has the iteration matrix* M *is:*

$$\rho(M)=\max_k\left\{|g(t_k)|,t_k\in T_{high}\right\}, \quad \textbf{(17)}$$

with: $T_{high}=\left\{k\,|\,k=c\frac{N}{2},...,\frac{N}{2}-1\right\}$ *-the* ***high frequencies domain*** *from the expansion and where:* $g(t_k)=\frac{c_k^{(m)}}{c_k^{(m-1)}},k=1,...,\frac{N}{2}-1$ *is the ratio of the coefficients of order* k *in the Fourier-transform expansion of the error,* $c\in(0,1)$ *being a fixed constant. The set of low frequencies is* $T_{low}=\left\{k\,|\,k=1,...,c\frac{N}{2}\right\}$.

Another reason we changed the definition (2) into (3) is the fact that in (7) the coefficients corresponding to different frequencies are complex numbers. And so the ratio of two succesive iterations of the same component can be made only in module. But when the expansion has all the terms in $\mathbb{R}$ the coefficients of each component will represent the amplitude of that oscillatory mode. In the remaining part of this section, we will determine the expression in the real number set of the Fourier expansion for the error.

The discrete Fourier expansion in a point $x_s=sh_l$ was (7):

$$E_s=\sum_{k=0}^{N-1}c_k e^{i\frac{2\pi k}{N}s} \quad \textbf{(18)}$$

with $N=n_l+1,s=0,...,N-1$.

Remark 2. *If we denote* $e_{k,s}=e^{i\frac{2\pi k}{N}s}$*, then:*

$$e_{N-k,s}=e^{i\frac{2\pi(N-k)}{N}s}=e^{i2\pi s}e^{-i\frac{2\pi k}{N}s}=\overline{e_{k,s}},\quad k,s=0,1,...,N-1.$$

Remark 3. *For* $N = n_l + 1 = 2^{l+1}$ *and* $k = \frac{N}{2}$ *we have* $e_{\frac{N}{2},s} = \cos(s\pi)$ *, thus* $e_{\frac{N}{2}}$ *is a real number. Also* $e_{0,s} = 1 \in \mathbb{R}$.

Using the above remarks, the relation (18) becomes:

$$E_s = c_0 + c_{\frac{N}{2}} e_{\frac{N}{2},s} + c_1 e_{1,s} + c_{N-1}\overline{e_{1,s}} + \ldots + c_{\frac{N}{2}-1} e_{\frac{N}{2}-1,s} + c_{\frac{N}{2}+1}\overline{e_{\frac{N}{2}-1,s}}. \quad \textbf{(19)}$$

The coefficients c_k can be computed using the reverse Fourier transform:

$$c_k = \frac{1}{N}\sum_{s=0}^{N-1} E_s e^{-i\frac{2\pi k}{N}s}, k = 0,1,\ldots,N-1. \quad \textbf{(20)}$$

From this equalty, using the remark that the values E_s are real numbers, it follows that:

Property 1. *The coefficients in the Fourier expansion for a real valued function have the properties:*

i. $c_{N-k} = \overline{c_k}, k = 0,1,\ldots,N-1;$

ii. $c_0 = \frac{1}{N}\sum_{s=0}^{N} E_s \in \mathbb{R};$

iii. $c_{\frac{N}{2}} \in \mathbb{R}.$

Proof **i.**

$$\begin{aligned} c_{N-k} &= \frac{1}{N}\sum_{s=0}^{N-1} E_s e^{-i\frac{2\pi(N-k)}{N}s} = \frac{1}{N}\sum_{s=0}^{N-1} E_s \overline{e_{N-k,s}} = \\ &= \frac{1}{N}\sum_{s=0}^{N-1} \overline{E}_s e_{k,s} = \frac{1}{N}\sum_{s=0}^{N-1} \overline{E_s \overline{e}_{k,s}} = \overline{c_k}, k = 0,1,\ldots,N-1 \end{aligned} \quad \textbf{(21)}$$

ii. Replacing in (20) $s = 0;$

iii. Using the ***Property 1a***: $c_{\frac{N}{2}} = c_{N-\frac{N}{2}} = \overline{c_{\frac{N}{2}}} \in \mathbb{R}$.

The relation (19) and ***Property 1a*** lead to the following expression of the error in a point:

$$\begin{aligned} E_s &= c_0 + c_{\frac{N}{2}}\cos(s\pi) + c_1 e_{1,s} + \overline{c}_1\overline{e_{1,s}} + \ldots + c_{\frac{N}{2}-1} e_{\frac{N}{2}-1,s} + \overline{c_{\frac{N}{2}-1}}\,\overline{e_{\frac{N}{2}-1,s}} = \\ &= c_0 + c_{\frac{N}{2}}\cos(s\pi) + 2Re\left(\sum_{k=1}^{N/2-1} c_k e_{k,s}\right) \in \mathbb{R} \end{aligned} \quad \textbf{(22)}$$

that has every term in $\mathbb{R}$. In this form, in the error expression are involved only half of the components used for the complex Fourier expansion.

If in (22) we write the complex coefficients as $c_k = a_k + ib_k$, the component of k order will have:

$$Re(c_k e_k) = a_k \cos\frac{2\pi ks}{N} - b_k \sin\frac{2\pi ks}{N}, k = 1,\ldots,\frac{N}{2}-1.$$

Thus:

$$E_s = c_0 + c_{\frac{N}{2}}\cos(\pi s) + 2\sum_{k=1}^{\frac{N}{2}-1}\left(a_k \cos\frac{2\pi ks}{N} - b_k \sin\frac{2\pi ks}{N}\right) = \\ = c_0 + c_{\frac{N}{2}}\cos(\pi s) - 2\sum_{k=1}^{\frac{N}{2}-1} A_k \sin\left(\frac{2\pi ks}{N} - \varphi\right) \tag{23}$$

only contains the frequencies: $k \in \left\{0,1,...,\frac{N}{2}-1\right\}$ and as the components are now real numbers, two successive iterative steps can be easily compared. Moreover, the coefficients $2A_k = 2\sqrt{a_k^2 + b_k^2}$ now represent the amplitude of the sinusoids composing the error.

From the complex expression of the amplification factor obtained for $k \geq 1$ ([12]):

$$c_k^{(m)} = c_k^{(m-1)} g(t_k), \quad m, k \geq 1, \tag{24}$$

we get the ratio of the amplitudes in this case:

$$\frac{A_k^{(m)}}{A_k^{(m-1)}} = \frac{\sqrt{\left(a_k^{(m)}\right)^2 + \left(b_k^{(m)}\right)^2}}{\sqrt{\left(a_k^{(m-1)}\right)^2 + \left(b_k^{(m-1)}\right)^2}} = |g_k(t)|, \quad k = 1,...,\frac{N}{2}-1.$$

Using the ***Definition 3***, the smoothing factor of the Gauss-Seidel method for problem (2) is:

$$\rho(M_{GS}) = \max\left\{\frac{|2\varepsilon - ah_l|}{\sqrt{(2\varepsilon + ah_l)^2 + (4\varepsilon)^2 - 8\varepsilon(2\varepsilon + ah_l)\cos t_k}}, t_k \in T_h\right\} = \\ = \frac{|2\varepsilon - ah_l|}{\sqrt{(2\varepsilon + ah_l)^2 + (4\varepsilon)^2 - 8\varepsilon(2\varepsilon + ah_l)\cos \pi c}}$$

and for the pondered Jacobi method:

$$\rho(M_{\omega J}) = \max_{k\in T_h} |g(t_k)| = \max_{k\in T_h} \sqrt{(1-\omega+\omega\cos t_k)^2 + \left(\frac{ah\omega}{2\varepsilon}\sin t_k\right)^2} = \\ = \sqrt{(1-\omega+\omega\cos \pi c)^2 + \left(\frac{ah\omega}{2\varepsilon}\sin \pi c\right)^2}.$$

Conclusions

In the following we present the results we obtained for the smoothing factor using the ***Definition 3*** for the Gauss-Seidel and pondered Jacobi methods. The data in **Tables 2**, **3 and 4**, presented below, lead to the following conclusions:

- The Jacobi method is not efficient as a smoother in the multigrid method due to the fact that it does not have the usual property of a numerical iterative method to efficiently reduce the high frequency error components, but it reduces only the middle part of the frequency spectrum
($k \in (0, N/2-1)$). And the multigrid method is based on the reduction of the high frequencies by the smoothing method. Moreover, the coarse grid correction, complementary to the smoothing method, is only efficient in reducing the low frequencies, because only these components (being smooth) can be well approximated on a coarse grid and in this case it should also reduce the frequencies from $k = \frac{N}{4}$ to $k = \frac{N}{2} - 1$ although these belong to T_{high}.

- On the other hand, for the Gauss-Seidel or pondered Jacobi method, the property of reducing the high frequencies determined using ***Definition 3*** is even better than was computed untill now using ***Definition 2*** for each iterative step and applying these methods more times makes them even more efficient.
- For the case of dominant convection (**Table 4**) as the number of layers used is growing, the amplification factor becomes smaller, thus it is better to use the numerical iterative method on a grid having more levels (at least six for the problem studied here) in order to have a reduction of the low frequencies components of the error, and even so the reduction is not efficient. This is why, as it is well known, for the convection-diffusion equation, when convection is dominant, the numerical iterative methods are often inefficient and special techniques have to be designed (for example stream-line diffusion [6], [7] or *hp*-multigrid methods [8], [10]) in order to overcome this inconvience.

$a=0, e=1$		$l=3$	$l=4$	$l=5$	$l=6$
c=0.25	GS	0.6786 0.8756	0.6786 0.9637	0.6786 0.9905	0.6786 0.9976
	J	0.9239 1.0000	0.9808 1.0000	0.9952 1.0000	0.9988 1.0000
	ωJ $(\omega = 0.5)$	0.8536 0.9619	0.8536 0.9904	0.8536 0.9976	0.8536 0.9994
c=0.5	GS	0.4472 0.8756	0.4472 0.9637	0.4472 0.9905	0.4472 0.9976
	J	0.9239 1.0000	0.9808 1.0000	0.9952 1.0000	0.9988 1.0000
	ωJ $(\omega = 0.5)$	0.5000 0.9619	0.5000 0.9904	0.5000 0.9976	0.5000 0.9994

Table 2: The smoothing factor of Gauss-Seidel (GS), Jacobi (J) and pondered Jacobi (ωJ) methods for model problem (2), $a = 0, \varepsilon = 1$ **-pure diffusion**

$a=1, e=1$		$l=3$	$l=4$	$l=5$	$l=6$
c=0.25	GS	0.6612 0.8656	0.6700 0.9620	0.6743 0.9903	0.6764 0.9976
	J	0.9240 1.0000	0.9808 1.0000	0.9952 1.0000	0.9988 1.0000
	ωJ $(\omega = 0.5)$	0.8536 0.9620	0.8536 0.9904	0.8536 0.9976	0.8536 0.9994
c=0.5	GS	0.4305 0.8656	0.4388 0.9620	0.4430 0.9903	0.4451 0.9976
	J	0.9240 1.0000	0.9808 1.0000	0.9952 1.0000	0.9988 1.0000
	ωJ $(\omega = 0.5)$	0.5002 0.9620	0.5001 0.9904	0.5000 0.9976	0.5000 0.9994

Table 3: The smoothing factor of Gauss-Seidel (GS) , Jacobi (J) and pondered Jacobi (ωJ) methods for model problem (2) and $a=1, \varepsilon=1$

$a=10, e=0.1$		$l=3$	$l=4$	$l=5$	$l=6$
c=0.25	GS	0.6950 0.8845	0.3088 0.7851	0.1497 0.7632	0.4308 0.9911
	J	3.1250 3.1250	1.5625 1.5625	0.9981 1.0000	0.9990 1.0000
	ωJ $(\omega = 0.5)$	1.6406 1.6406	1.0167 1.0167	0.8971 0.9983	0.8647 0.9994
c=0.5	GS	0.4635 0.8845	0.1730 0.7851	0.0817 0.7632	0.2502 0.9911
	J	3.1250 3.1250	1.5625 1.5625	0.9981 1.0000	0.9990 1.0000
	ωJ $(\omega = 0.5)$	1.6406 1.6406	0.9276 1.0167	0.6345 0.9983	0.5368 0.9994

Table 4: The smoothing factor of Gauss-Seidel (GS) , Jacobi (J) and pondered Jacobi (ωJ) methods for model problem (2) and $a=10, \varepsilon=0.1$ **-dominant convection**

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