

# A REMARKABLE PROPERTY OF EXCESSIVE FUNCTIONS CONE

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**Abstract:** In this paper we present remarkable properties of excessive functions with respect to absolute continue resolvent. The cone of above functions forms a  $H$ -cone.

**Keywords:** excessive function,  $H$ -cone, absolute continue resolvent

The excessive functions with respect to absolute continue resolvent.

**Definiton 1.** If  $(X, \mathcal{B})$  is a measurable space and  $p\mathcal{B}$  the numerical positive measurable functions on  $E$  we denote a kernel on  $(X, \mathcal{B})$  a map  $V: p\mathcal{B} \rightarrow p\mathcal{B}$  with properties:

1.  $V0 = 0$
2.  $V(\sum_n f_n) = \sum_n V(f_n)$ .

**Definition 2.** A family  $\mathcal{V} = (V_\alpha)_{\alpha > 0}$  of kernels on measurable space  $(X, \mathcal{B})$  is called resolvent if the followings hold:

1.  $V_\alpha V_\beta = V_\beta V_\alpha, \forall \alpha, \beta > 0$
2.  $V_\alpha = V_\beta + (\beta - \alpha)V_\alpha V_\beta, \forall \alpha, \beta > 0, \alpha < \beta$ .

The resolvent is called sub-Markovian if for any  $\alpha > 0$  we have  $\alpha V_\alpha \leq 1$ .

The kernel  $V = \sup_\alpha V_\alpha f$  is called initially kernel.

**Definition 3.** A map  $s \in p\mathcal{B}$  is called  $\mathcal{V}$ -excessive if the followings hold:

1.  $s$  is  $\mathcal{V}$ -supermedian, i.e  $\alpha V_\alpha s \leq s$  for any  $\alpha > 0$
2.  $\sup_\alpha \alpha V_\alpha s = s$
3.  $s$  is finite  $\mathcal{V}$  a.e. (a set  $A \subset X$  is  $\mathcal{V}$ -negligible if there exists  $A' \subset \mathcal{B}$  such that  $A \subset A'$  and  $V_\alpha(A') = 0$ , for any  $\alpha > 0$ ).

**Proposition 1.** We have the following properties:

1. If  $s \in \mathcal{S}_\mathcal{V}$  it follows that  $\hat{s} = \sup_\alpha V_\alpha s \in \mathcal{S}_\mathcal{V}, \hat{s} \leq s$
2. If  $s, t \in \mathcal{S}_\mathcal{V}$  it follows that  $\widehat{s+t} = \hat{s} + \hat{t}$
3. For any  $(s_n)_n \subset \mathcal{S}_\mathcal{V}, s_n \uparrow s$  it follows that  $s_n \uparrow s$
4. For any  $s \in \mathcal{S}_\mathcal{V}$  it follows that  $V_\alpha \hat{s} = V_\alpha s$
5. If  $s \in \mathcal{S}_\mathcal{V} \Rightarrow \hat{s} = s$   $\mathcal{V}$ - a.p.t.
6. If  $s \in \mathcal{S}_\mathcal{V} \Rightarrow \hat{\hat{s}} = s$  see (e.g [1] Proposition 1.18)

**Theorem 1.** We have the following properties:

1. For any  $t \in \mathcal{E}_\mathcal{V}$  (the set of excessive functions),  $\alpha, \beta > 0$  it follows that  $\alpha s + \beta t \in \mathcal{E}_\mathcal{V}$

2. For any  $s, t \in \mathcal{E}_V, s \leq t$   $\mathcal{V}$ -a. e. it follows that  $s \leq t$
3. For any  $(s_n)_n \subseteq \mathcal{E}_V$  there exists  $\bigwedge_n s_n$  and we have  $\bigwedge_n s_n = \overline{\inf} s_n$  and  $s + \bigwedge_n s_n = \bigwedge_n (s + s_n)$ , for any  $s \in \mathcal{E}_V$
4. For any  $(s_n)_n \subset \mathcal{E}_V$  dominated there exists  $\bigvee_n s_n$  and we have  $\bigvee_n s_n = R^V(\sup_n s_n)$ , where  $R^V f = \inf\{s \in \mathcal{S}_V \mid s \geq f\}$  ( $s \in \mathcal{S}_V$  if and only if  $\alpha \bigvee s \leq s$ , for any  $\alpha > 0$ ) (see e.g. [1] Theorem 1.1.9)  
If  $(s_n)_n$  is an increasing family we have  $\bigvee_n s_n = \sup_n s_n$ .
5. For any  $s, t, u \in \mathcal{E}_V$  such that  $s \leq t + u$ , there exists  $s', s'' \in \mathcal{E}_V$  such that  $s' + s''$  and  $s' \leq t, s'' \leq u$ . (see e.g. [4] Theorem 1.1.9)

**Definition 4.** A resolvent  $\mathcal{V} = (V_\alpha)_{\alpha > 0}$  is absolutely continue (with respect to a finite measure  $m$ ) if for any  $f \in \mathcal{F}$ , such that  $\int f dm = 0$  it follows that  $V_\alpha f = 0$ , for any  $\alpha > 0$ ).

**Theorem 2.** Let  $\mathcal{V} = (V_\alpha)_{\alpha > 0}$  an absolutely continue resolvent (with respect to a finite measure  $m$ ). Then the followings hold:

1. For any increasingly and dominated family  $(s_i)_{i \in I} \subset \mathcal{E}_V$  we have  $\sup_{i \in I} s_i \in \mathcal{E}_V$  and there exists an increasing family  $(s_{i_n})_{i_n \in I'}$ , such that  $\sup_n s_{i_n} = \sup_{i \in I} s_i = \bigvee_{i \in I} s_i$ .
2. For any family  $(s_i)_{i \in I} \in \mathcal{E}_V$  there exists a subsequence  $(i_n)_n \subset I$ , such that  $\bigwedge_{i \in I} s_i = \overline{\inf} s_{i_n}$  and  $s + \bigwedge_{i \in I} s_i = \bigwedge_{i \in I} (s + s_i)$ , for any  $s \in \mathcal{E}_V$  (see e.g [4] Theorem 1.1.10)

**Definition 4.** An ordered convex cone  $S$  is called  $H$ -cone if the following axioms are satisfied:

1. For any non-empty family  $A \subset S$  there exists  $\bigwedge A$  and we have  $s + \bigwedge A = \bigwedge (s + A)$ , for any  $s \in S$ ;
2. For any increasing and dominated family  $A \subset S$  there exists  $\bigvee A$  and we have  $\bigvee (A + u) = \bigvee A + u$ , for any  $u \in S$ ;
3.  $S$  satisfies the Riesz decomposition property i.e. for any  $s, s_1, s_2 \in S$  such that  $s \leq s_1 + s_2$  there exist  $t_1, t_2 \in S$  satisfying  $s = t_1 + t_2, t_1 \leq s_1, t_2 \leq s_2$ .

From two theorem above it follows that the cone of excessive functions with respect to absolute continue resolvent forms a  $H$ -cone.

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